Classical Mechanics & Special Relativity for Starters

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Contents

1. Introduction

Updated: 04 feb 2026 This book provides an introduction for freshman students into the world of classical mechanics and special relativity theory. Much of physics is build on the basic ideas from classical mechanics. Hence an early introduction to the topic can be beneficial for new students. However, at the start of studying physics, lots of the required math is not available yet. That means that all kind of concepts that are very useful cannot be invoked in the lectures and teaching. That does not have to be a disadvantage. It can also be used to help the students by introducing some math and coupling it directly to the physics, making more clear why mathematics should be studied and what its 'practical use' is. With this book, we aim to introduce new students directly at the start of their studies into the world of physics, more specifically the world of Newton, Galilei and many others who laid the foundation of physics. We aim to help students getting a good understanding of the theory, i.e. the framework of physics. What is 'work' and why do we use it? Why is kinetic energy $\frac{1}{2}mv^2$ and not $\frac{1}{3}mv^2$ or $\frac{1}{2}mv^3$? Both 3′s are fundamentally wrong, but what is behind it?

1.1 About this book

Classical mechanics is the starting point of physics. Over the centuries, via Newton's three fundamental laws formulated around 1687, we have built a solid framework describing the material world around us. On these pages, you will find a textbook with animations, demos, interactive elements and exercises for studying introductory classical mechanics. Moreover, we will consider the first steps of Einstein's Special Theory of Relativity published 1905.

This material is made to support first year students from the BSc Applied Physics at Delft University of Technology during their course *Classical Mechanics and Relativity Theory*, MechaRela for short. But, of course, anybody interested in Classical Mechanics and Special Relativity is more than welcome to use this book.

With this e-book our aim is to provide learning material that is:

- self-contained
- easy to modify and thus improve over the years
- interactive by providing demos, interactive elements and exercises next to the lectures

This book is based on Mudde & Rieger 2025.

That book was already beyond introductory level and presumed a solid basis in both calculus and basic mechanics. All texts in this book were revised, additional examples and exercises were included, picture and drawings have been updated and interactive materials have been included. Hence, this book should be considered a stand-alone new version. Note that we made good use of other open educational resources, several exercises stem from such resources. Where we use external materials, we acknowledge and credit the original sources.

1.1.1 Special features

In this book you will find some 'special' features. Some of these are indicated by their own formatting:

Intermezzos

Intermezzos contain background information on the topic or on the people that worked on relevant concepts.

Experiments

We include some basic experiments that can be done at home.

Example: Examples

We provide various examples showcasing, e.g., calculations.

Exercise 1.1: 🌶

Each chapter includes a variety of exercises tailored to the material. We distinguish between exercises embedded within the instructional text and those presented on separate pages. The in-text exercises should be completed while reading, as they offer immediate feedback on whether the concepts and mathematics are understood. The separate exercise sets are intended for practice after reading the text and attending the lectures.

To indicate the level of difficulty, each exercise is marked with 1, 2, or 3 🌶

Python

We include in-browser python code, as well as downloadable .py files which can be executed locally. If there is an in-browser, press the ON-button to 'enable compute'. Try it by pushing the ON-button and subsequently the play button and see the output in de code-cell below.

```
print("The square root of 2 is: ", 2 ** 0.5)
```

The interactive elements, such as Python code, hover over functionality, embedded youtube clips etc, only work in the online environment, not in the pdf or printed book. Where possible we included qr codes and links to the online clips. We also include interactive exercises made with Grasple. These exercises provide immediate feedback on your answers, allowing you to learn from your mistakes and deepen your understanding of the material. Here is an example exercise:

Grasple 1: grasple example

New concepts, such as *Free body diagram*, are introduced with a hover-over. If you move your mouse over the italicized part of the text, you will get a short explanation.

You have the opportunity to download some of the materials as Jupyter Notebook file and play with the code offline. We advise you to use Jupyter Lab in combination with MyST.

1.1.2 Feedback

This is the first version (second cycle) of this book. Although many have worked on it and several iterations have been made, there might still be issues. Do you see a mistake? Do you have suggestions for exercises? Are parts missing or abundant? Tell us! You can use the `Feedback` button in the top right of the screen. You will need a (free) GitHub account to report an issue!

1.2 Authors

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Special thanks to Hanna den Hertog for (re)making most of the drawings, Luuk Fröling for his technical support and Dion Hoeksema for converting the .js scripts to .py files. Also thanks to Vebe Helmes, Alexander Lopes-Cardozo, Sep Schouwenaar, Alesja Zorina, Winston de Greef and Boas Bakker for their comments and suggestions.

1.3 Open Educational Resource

This website is a Jupyter Book. Source files are available for download using the button on the top right.

1.3.1 Software and license

This website is a Jupyter Book. Markdown source files are available for download using the button on the top right, licensed under CC-BY-NC (unless stated otherwise). All python codes / apps are freely reusable, adaptable and redistributable (CC0).

1.3.2 Images, videos, apps, intermezzos

The cover image is inspired by the work of 3Blue1Brown developer Grant Sanderson.

All vector images have been made by Hanna den Hertog, and are available in vector format through the repository. For reuse, adapting and redistribution, adhere to the CC-BY licences.

We embedded several clips from 3Blue1Brown in accord with their licences requirements.

The embedded vpython apps are made freely available from trinket.

Some videos from NASA are included, where we adhere to their regulations.

At various places we use pictures which are in the public domain. We comply to the regulations with regard to references.

The Intermezzos, which elaborate on the lives of various scientists and the efforts behind key physical discoveries, are composed by drawing from a range of different sources. Rather than directly reproducing any one account, these stories have been reworked into a narrative that fits the context and audience of this book.

1.3.3 How to cite this book

R.F. Mudde, B. Rieger, C.F.J. Pols, *Classical Mechanics & Special Relativity for Beginners*, CC BY-NC

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@book{MuddeRiegerPols2025,
  author    = {Robert F. Mudde and Bernd Rieger and Freek Pols},
  title     = {Classical Mechanics \& Special Relativity for Beginners},
  year      = {2025},
  publisher = {TU Delft Open},
  note      = {CC BY-NC},
  url       = {https://interactivetextbooks.tudelft.nl/mecharela}
}
```

2. Mechanics

Updated: 04 feb 2026 In this part we cover the fundamentals of Classical Mechanics. We discuss the three laws of Newton and their first consequences. This part focusses on the primary concepts and quantities: *Force, Work, Energy, Angular Momentum.* We derive and discuss the conservation equations of these and their applications. Two topics receive special attention: *Oscillations* and *Collisions.* We restrict the discussion to one-dimensional cases as much as possible to help understand the physics and not get lost in multi-dimensional problems at an early stage. However, more-dimensionality is not avoided as, for instance, it should be clear from the start that physics not only deals with numbers (better: scalars) but equally important, if not more important, with vectors. Moreover, *angular momentum* and *torque* by their very nature require multi-dimensional thinking.

There are also subjects that we don't touch upon. We will not deal with rigid bodies (although some of the ideas are met when talking about kinetic energy: its translational versus rotational flavors). Rigid bodies require a higher level of abstract thinking and will take up quite some time that is not available in most introductory courses on Classical Mechanics. Nor will we discuss non-inertial frames of reference and fictitious forces like the centrifugal and Coriolis Force. This is left for later years. Finally, the concepts of the Lagrangian and Hamiltonian are left for an advanced course in Classical Mechanics.

2.1 The language of Physics

Updated: 04 feb 2026 Physics is the science that seeks to understand the fundamental workings of the universe: from the motion of everyday objects to the structure of atoms and galaxies. To do this, physicists have developed a precise and powerful language: one that combines mathematics, colloquial and technical language, and visual representations. This language allows us not only to describe how the physical world behaves, but also to predict how it will behave under new conditions.

In this chapter, we introduce the foundational elements of this language, covering how to express physical ideas using equations, graphs, diagrams, and words. You'll also get a first taste of how physics uses numerical simulations as an essential complement to analytical problem solving.

This language is more than just a set of tools—it is how physicists *think*. Mastering it is the first step in becoming fluent in physics.

2.1.1 Representations

Physics problems and concepts can be represented in multiple ways, each offering a different perspective and set of insights. The ability to translate between these representations is one of the most important skills you will develop as a physics student. In this section, we examine three key forms of representation: equations, graphs and drawings, and verbal descriptions using the context of a base jumper, see Figure 1.

Figure 2.1: A base jumper is used as context to get familiar with representation, picture from https://commons.wikimedia.org/wiki/File:04SHANG4963.jpg

2.1.1.1 Verbal descriptions

Words are indispensable in physics. Language is used to describe a phenomenon, explain concepts, pose problems and interpret results. A good verbal description makes clear:

- What is happening in a physical scenario;
- What assumptions are being made (e.g., frictionless surface, constant mass);
- What is known and what needs to be found.

Example: Base jumper: Verbal description

Let us consider a base jumper jumping from a 300 m high building. We take that the jumper drops from that height with zero initial velocity. We will assume that the stunt is performed safely and in compliance with all regulations/laws. Finally, we will assume that the problem is 1-dimensional: the jumper drops vertically down and experiences only gravity, buoyancy and air-friction.

We know (probably from experience) that the jumper will accelerate. Picking up speed increases the drag force acting on the jumper, slowing the *acceleration* (meaning it still accelerates!). The speed keeps increasing until the jumper reaches its terminal velocity, that is the velocity at which the drag (+ buoyancy) exactly balance gravity and the sum of forces on the jumper is zero. The jumper no longer accelerates.

Can we find out what the terminal velocity of this jumper will be and how long it takes to reach that velocity?

2.1.1.2 Visual representations

Visual representations help us interpret physical behavior at a glance. Graphs, motion diagrams, free-body diagrams, and vector sketches are all ways to make abstract ideas more tangible.

- **Drawings** help illustrate the situation: what objects are involved, how they are moving, and what forces act on them.
- **Graphs** (e.g., position vs. time, velocity vs. time) reveal trends and allow for estimation of slopes and areas, which have physical meanings like velocity and displacement.

Example: Base jumper: Free body diagram

The situation of the base jumper is sketched in Figure 2 using a Free body diagram. Note that all details of the jumper are ignored in the sketch.

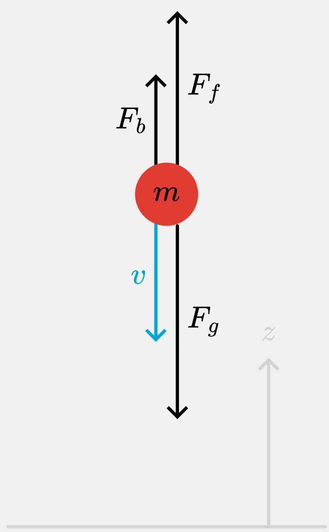


Figure 2.2: Force acting on the jumper.

- m = mass of jumper (in kg);
- v = velocity of jumper (in m/s);
- F_g = gravitational force (in N);
- F_f = drag force by the air (in N);
- F_b = buoyancy (in N): like in water also in air there is an upward force, equal to the weight of the displaced air.

2.1.1.3 Equations

Equations are the compact, symbolic expressions of physical relationships. They tell us how quantities like velocity, acceleration, force, and energy are connected.

Example: Base jumper: equations

The forces acting on the jumper are already shown in Figure 2. Balancing of forces tells us that the jumper might reach a velocity such that the drag force and buoyancy exactly balance gravity and the jumper no longer accelerates:

$$F_g = F_f + F_b \tag{2.1}$$

We can specify each of the forces:

$$\begin{aligned} F_g &= -mg = -\rho_p V_p g \\ F_f &= \frac{1}{2}\rho_{air} C_D A v^2 \\ F_b &= \rho_{air} V_p g \end{aligned} \tag{2.2}$$

with g the acceleration of gravity, ρ_p the density of the jumper ($\approx 10^3\ \mathrm{kg/m^3}$), V_p the volume of the jumper, ρ_{air} the density of air ($\approx 1.2\ \mathrm{kg/m^3}$), C_D the so-called drag coefficient, A the frontal area of the jumper as seen by the air flowing past the jumper.

A physicist is able to switch between these representations, carefully considering which representations suits best for the given situation. We will practice these when solving problems.

Danger

Note that in the example above we neglected directions. In our equations we should have been using vector notation, which we will cover in one of the next sections in this chapter.

2.1.2 How to solve a physics problem?

One of the most common mistakes made by 'novices' when studying problems in physics is trying to jump as quickly as possible to the solution of a given problem or exercise by picking an equation and slotting in the numbers. For simple questions, this may work. But when stuff gets more complicated, it is almost a certain route to frustration.

There is, however, a structured way of problem solving, that is used by virtually all scientists and engineers. Later this will be second nature to you, and you will apply this way of working automatically. It is called IDEA, an acronym that stands for:

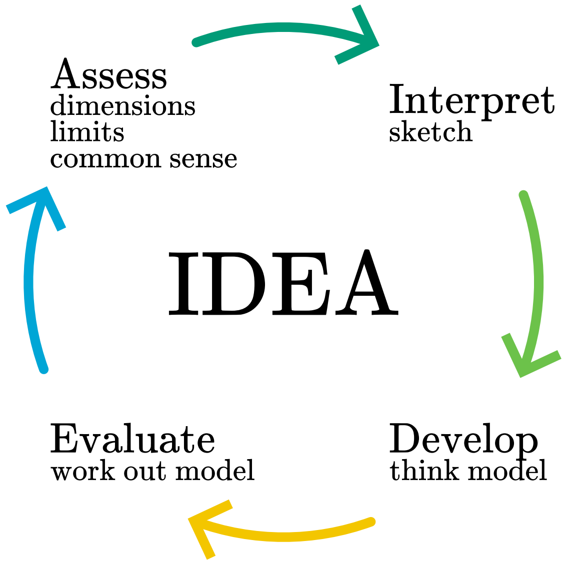


Figure 2.3: IDEA

- **Interpret** - First think about the problem. What does it mean? Usually, making a sketch helps. ***Actually:*** *always start with a sketch*;
- **Develop** - Build a model, from coarse to fine, that is, first think in the governing phenomena and then subsequently put in more details. Work towards writing down the equation of motion and boundary conditions;
- **Evaluate** - Solve your model, i.e. the equation of motion;
- **Assess** - Check whether your answer makes any sense (e.g. units OK? What order of magnitude did we expect?). Is our answer in the order of magnitude that we expected?

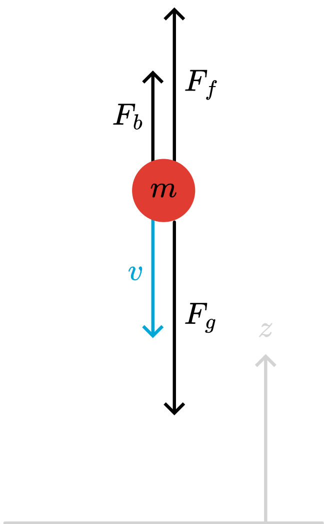
We will practice this and we will see that it actually is a very relaxed way of working and thinking. We strongly recommend to apply this strategy for your homework and exams (even though it seems strange in the beginning).

The first two steps (Interpret and Develop) typically take up most of the time spend on a problem.

2.1.2.1 Example

Interpret

Three forces act on the jumper, shown in the figure below. Finding the terminal velocity implies that all forces are balanced ($\sum F = 0$).



The buoyancy force is much smaller than the force of gravity (about 0.1%) and we neglect it.

Develop

We know all forces: gravitational force equals the drag force

$$
\begin{aligned}
F_g &= F_f \\
mg &= \frac{1}{2}\rho_{air} C_D A v^2
\end{aligned}
\tag{2.3}
$$

Evaluate

Assume a mass of 75 kg, an acceleration due to gravity of 9.81 $\mathrm{m/s^2}$, and air density of 1.2 $\mathrm{kg/m^3}$, a drag coefficient of 1, a frontal surface area of 0.7 $\mathrm{m^2}$.

$$
mg = \frac{1}{2}\rho_{air} C_D A v^2 \tag{2.4}
$$

Rewriting:

$$
\begin{aligned}
v &= \sqrt{\frac{2mg}{\rho_{air} C_D A}} \\
v &= \sqrt{\frac{2 \cdot 75\ (\mathrm{kg}) \cdot 9.81\ (\mathrm{m/s^2})}{1.2\ (\mathrm{kg/m^3}) \cdot 1 \cdot 0.7\ (\mathrm{m^2})}} \\
v &= 40\ \mathrm{m/s}
\end{aligned}
\tag{2.5}
$$

Assess

We may know that raindrops typically reach a terminal velocity of less than 10 m/s. A terminal velocity of 40 m/s seems therefore plausible for a much heavier object.

Note that we didn't solve the problem entirely! We only calculated the terminal velocity, where the question was how long it would roughly take to reach such a velocity.

Good Practice

It is a good habit to make your mathematical steps small: one-by-one. Don't make big jumps or do multiple steps at once. If you make a mistake, it will be very hard to trace it back.

Next: check always the dimensional correctness of your equations: that is easy to do and you will find the majorities of your mistakes.

Finally, use letters to denote quantities, including π. The reason for this is:

- letters have meaning and you can easily put dimensions to them;
- letters are more compact;
- your expressions usually become easier to read and characteristic features of the problem at hand can be recognized.

Powers of ten

In physics, powers of ten are used to express very large or very small quantities compactly and clearly, from the size of atoms ($\approx 10^{-10}$ m) to the distance between stars ($\approx 10^{16}$ m). This notation helps compare scales, estimate orders of magnitude, and maintain clarity in calculations involving extreme values.

We use prefixes to denote these powers of ten in front of the standard units, e.g. km for 1000 meters, ms for milliseconds, GB for gigabyte that is 1 billionbytes. Here is a list of prefixes.

Prefix	Symbol	Math	Prefix	Symbol	Math
Yocto	y	10^{-24}	Base	•	10^{0}
Zepto	z	10^{-21}	Deca	da	10^{1}
Atto	a	10^{-18}	Hecto	h	10^{2}
Femto	f	10^{-15}	Kilo	k	10^{3}
Pico	p	10^{-12}	Mega	M	10^{6}
Nano	n	10^{-9}	Giga	G	10^{9}
Micro	µ	10^{-6}	Tera	T	10^{12}
Milli	m	10^{-3}	Peta	P	10^{15}
Centi	c	10^{-2}	Exa	E	10^{18}
Deci	d	10^{-1}	Zetta	Z	10^{21}
Base	•	10^{0}	Yotta	Y	10^{24}

On quantities and units

Each quantity has a unit. As there are only so many letters in the alphabet (even when including the Greek alphabet), letters are used for multiple quantities. How can we distinguish then meters from mass, both denoted with the letter ‘m’? Quantities are expressed in italics (m) and units are not (m).

We make extensively use of the International System of Units (SI) to ensure consistency and precision in measurements across all scientific disciplines. The seven base SI units are:

Unit	Symbol	Quantity
meter	m	length
kilogram	kg	mass
second	s	time
ampere	A	electric current
kelvin	K	temperature
mole	mol	amount of substance
candela	cd	luminous intensity

All other quantities can be derived from these using dimension analysis:

$$\begin{aligned} W &= F \cdot s = ma \cdot s = m \frac{\Delta v}{\Delta t} \cdot s \\ &= [\mathrm{N}] \cdot [\mathrm{m}] = [\mathrm{kg}] \cdot [\mathrm{m/s^2}] \cdot [\mathrm{m}] = [\mathrm{kg}] \cdot \frac{[\mathrm{m/s}]}{[\mathrm{s}]} \cdot [\mathrm{m}] = \left[\frac{\mathrm{kgm^2}}{\mathrm{s^2}}\right] \end{aligned} \tag{2.6}$$

Note: Newton is the person, fully written the unit N is newton, without capitalization of the first letter.

Tip

For a more elaborate description of quantities, units and dimension analysis, see the manual of the first year physics lab course.

2.1.3 Calculus

Most of the undergraduate theory in physics is presented in the language of Calculus. We do a lot of differentiating and integrating, and for good reasons. The basic concepts and laws of physics can be cast in mathematical expressions, providing us the rigor and precision that is needed in our field. Moreover, once we have solved a certain problem using calculus, we obtain a very rich solution, usually in terms of functions. We can quickly recognize and classify the core features that help us understand the problem and its solution much deeper.

Given the example of the base jumper, we would like to know the jumper's position as a function of time. We can answer this question by applying Newton's second law (though it is covered in secondary school, the next chapter explains in detail Newton's laws of motion):

$$\sum F = F_g - F_f = ma = m\frac{dv}{dt} \tag{2.7}$$

$$m\frac{dv}{dt} = mg - \frac{1}{2}\rho_{air}C_D A v^2 \tag{2.8}$$

Clearly this is some kind of differential equation: the change in velocity depends on the velocity itself ($\frac{dv}{dt} = ..v(t)$). Before we even try to solve this problem ($v(t) = ...$), we have to dig deeper in the precise notation, otherwise we will get lost in directions and sign conventions.

2.1.3.1 Differentiation

Many physical phenomena are described by differential equations. That may be because a system evolves in time, or because it changes from location to location. In our treatment of the principles of classical mechanics, we will use differentiation with respect to time a lot. The reason is obviously found in Newton's 2^{nd} law: $F = ma$.

The acceleration a is the derivative of the velocity with respect to time ($a = \frac{dv}{dt}$); velocity in itself is the derivative of position with respect to time ($v = \frac{dx}{dt}$). Or when we use the equivalent formulation with momentum: $\frac{dp}{dt} = F$. So, the change of momentum in time is due to forces. Again, we use differentiation, but now of momentum.

There are three common ways to denote differentiation. The first one is by 'spelling it out':

$$v = \frac{dx}{dt} \text{ and } a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{2.9}$$

- Advantage: it is crystal clear what we are doing.
- Disadvantage: it is a rather lengthy way of writing.

Newton introduced a different flavor: he used a dot above the quantity to indicate differentiation with respect to time. So,

$$v = \dot{x}, \text{ or } a = \dot{v} = \ddot{x} \tag{2.10}$$

- Advantage: compact notation, keeping equations compact.
- Disadvantage: a dot is easily overlooked or disappears in the writing.

Finally, in math often the prime is used: $\frac{df}{dx} = f'(x)$ or $\frac{d^2f}{dx^2} = f''(x)$. Similar advantage and disadvantage as with the dot notation.

Important

$$v = \frac{dx}{dt} = \dot{x} = x' \tag{2.11}$$

$$a = \frac{dv}{dt} = \dot{v} = \frac{d^2x}{dt^2} = \ddot{x} \tag{2.12}$$

It is just a matter of notation.

2.1.4 Definition of velocity, acceleration and momentum

In mechanics, we deal with forces on particles. We try to describe what happens to the particles, that is, we are interested in the position of the particles, their velocity and acceleration. We need a formal definition, to make sure that we all know what we are talking about.

1-dimensional case

In one dimensional problems, we only have one coordinate to take into account to describe the position of the particle. Let's call that x. In general, x will change with time as particles can move. Thus, we write $x(t)$ showing that the position, in principle, is a function of time t. How fast a particle changes its position is, of course, given by its velocity. This quantity describes how far an object has traveled in a given time interval: $v = \frac{\Delta x}{\Delta t}$. However, this definition gives actually the average velocity in the time interval Δt. The (momentary) velocity is defined as:

Definition Velocity

$$v \equiv \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{(t + \Delta t) - t} = \frac{dx}{dt} \tag{2.13}$$

Note that we here use $\equiv$ rather than $=$ to indicate that this is a definition.

Similarly, we define the acceleration as the change of the velocity over a time interval Δt: $a = \frac{\Delta v}{\Delta t}$. Again, this is actually the average acceleration and we need the momentary one:

Definition Acceleration

$$a \equiv \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{(t + \Delta t) - t} = \frac{dv}{dt} \tag{2.14}$$

Consequently,

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \tag{2.15}$$

Now that we have a formal definition of velocity, we can also define momentum: momentum is mass times velocity, in math:

Definition Momentum

$$p \equiv mv = m\frac{dx}{dt} \tag{2.16}$$

In the above, we have not worried about how we measure position or time. The latter is straight forward: we use a clock to account for the time intervals. To find the position, we need a ruler and a starting point from where we measure the position. This is a complicated way of saying the we need a coordinate system with an origin. But once we have chosen one, we can measure positions and using a clock measure changes with time.

Figure 2.5: Calculating velocity requires both position and time, both easily measured e.g. using a stopmotion where one determines the position of the car per frame given a constant time interval.

2.1.4.1 Vectors - more dimensional case

Position, velocity, momentum, force: they are all *vectors*. In physics we will use vectors a lot. It is important to use a proper notation to indicate that you are working with a vector. That can be done in various ways, all of which you will probably use at some point in time. We will use the position of a particle located at point P as an example.

Tip

See the linear algebra book on vectors.

Position vector

We write the position **vector** of the particle as $\vec{r}$. This vector is a 'thing', it exists in space independent of the coordinate system we use. All we need is an origin that defines the starting point of the vector and the point P, where the vector ends.

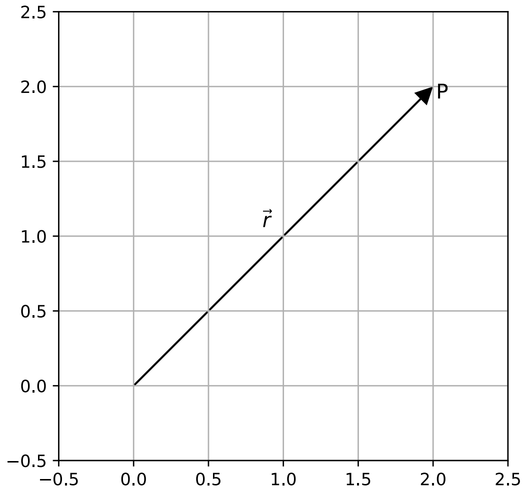


Figure 2.6: Some physical quantities (velocity, force etc) can be represented as a vector. The have in common the direction, magnitude and point of application.

A coordinate system allows us to write a representation of the vector in terms of its coordinates. For instance, we could use the familiar Cartesian Coordinate system {x,y,x} and represent $\vec{r}$ as a column.

$$\vec{r} \rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.17}$$

Alternatively, we could use unit vectors in the x, y and z-direction. These vectors have unit length and point in the x, y or z-direction, respectively. They are denoted in varies ways, depending on taste. Here are 3 examples:

$$\begin{aligned} &\hat{x},\ \hat{\imath},\ \vec{e}_x \\ &\hat{y},\ \hat{\jmath},\ \vec{e}_y \\ &\hat{z},\ \hat{k},\ \vec{e}_z \end{aligned} \tag{2.18}$$

With this notation, we can write the position vector $\vec{r}$ as follows:

$$\begin{aligned} \vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} \\ \vec{r} &= x\hat{\imath} + y\hat{\jmath} + z\hat{k} \\ \vec{r} &= x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \end{aligned} \tag{2.19}$$

Note that these representations are equivalent: the difference is in how the unit vectors are named. Also note, that these three representations are all given in terms of vectors. That is important to realize: in contrast to the column notation, now all is written at a single line. But keep in mind: $\hat{x}$ and $\hat{y}$ are perpendicular **vectors**.

Other textbooks

Note that other textbooks may use bold symbols to represent vectors:

$$\vec{F} = m\vec{a} \tag{2.20}$$

is the same as

$$\boldsymbol{F} = m\boldsymbol{a} \tag{2.21}$$

Differentiating a vector

We often have to differentiate physical quantities: velocity is the derivative of position with respect to time; acceleration is the derivative of velocity with respect to time. But you will also come across differentiation with respect to position ($\frac{d}{dx}$).
As an example, let's take velocity. Like in the 1-dimensional case, we can ask ourselves: how does the position of an object change over time? That leads us naturally to the definition of velocity: a change of position divided by a time interval:

Definition Velocity (Vector)

$$\vec{v} \equiv \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt} \tag{2.22}$$

What does it mean? Differentiating is looking at the change of your 'function' when you go from x to $x + dx$:

$$\frac{df}{dx} \equiv \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{2.23}$$

In 3 dimensions we will have that we go from point P, represented by $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ to 'the next point' $\vec{r} + d\vec{r}$. The small vector $d\vec{r}$ is a small step forward in all three directions, that is a bit dx in the x-direction, a bit dy in the y-direction and a bit dz in the z-direction. Consequently, we can write $\vec{r} + d\vec{r}$ as

$$\begin{aligned} \vec{r} + d\vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} + dx\hat{x} + dy\hat{y} + dz\hat{z} \\ &= (x + dx)\hat{x} + (y + dy)\hat{y} + (z + dz)\hat{z} \end{aligned} \tag{2.24}$$

Now, we can take a look at each component of the position and define the velocity component as, e.g., in the x-direction

$$v_x = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt} \tag{2.25}$$

Applying this to the 3-dimensional vector, we get

$$\begin{aligned} \vec{v} \equiv \frac{d\vec{r}}{dt} &= \frac{d}{dt}(x\hat{x} + y\hat{y} + z\hat{z}) \\ &= \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} \\ &= v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \end{aligned} \tag{2.26}$$

Note that in the above, we have used that according to the product rule:

$$\frac{d}{dt}(x\hat{x}) = \frac{dx}{dt}\hat{x} + x\frac{d\hat{x}}{dt} = \frac{dx}{dt}\hat{x} \tag{2.27}$$

since $\frac{d\hat{x}}{dt} = 0$ (the unit vectors in a Cartesian system are constant). This may sound trivial: how could they change; they have always length 1 and they always point in the same direction. Trivial as this may be, we will come across unit vectors that are not constant as their direction may change. But we will worry about those examples later.

Now that the velocity of an object is defined, we can introduce its momentum:

Definition Momentum (Vector)

$$\vec{p} \equiv m\vec{v} = m\frac{d\vec{r}}{dt} \tag{2.28}$$

Albeit we have now a formal definition of momentum, we come later to its physical interpretation.

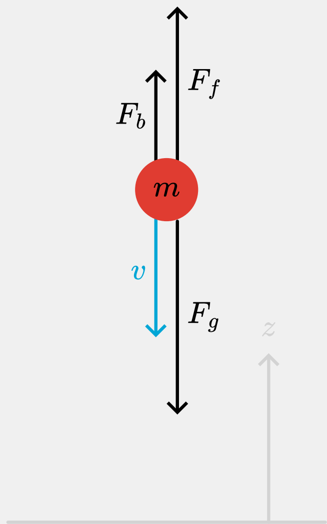
We can use the same reasoning and notation for acceleration:

Definition Acceleration (Vector)

$$\vec{a} \equiv \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \tag{2.29}$$

Example: The base jumper

Given the above explanation, we can now reconsider our description of the base jumper.



We see a z-coordinate pointing upward, where the velocity. As gravitational force is in the direction of the ground, we can state

$$\vec{F}_g = -mg\hat{z} \tag{2.30}$$

Buoyancy is clearly along the z-direction, hence

$$\vec{F}_b = \rho_{air} V g \hat{z} \tag{2.31}$$

The drag force is a little more complicated as the direction of the drag force is always against the direction of the velocity $-\vec{v}$. However, in the formula for drag we have v^2. To solve this, we can write

$$\vec{F}_f = -\frac{1}{2}\rho_{air} C_D A \mid v \mid \vec{v} \tag{2.32}$$

Note that $\hat{z}$ is missing in (32) on purpose. That would be a simplification that is valid in the given situation, but not in general.

2.1.5 Numerical computation and simulation

In simple cases we can obtain a physical model where we can derive an analytical solution. In the case of the base jumper, an analytical solution exists, though it is not trivial and requires some advanced operations as separation of variables and partial fractions:

$$v(t) = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}} t\right) \tag{2.33}$$

with

$$k = \frac{1}{2}\rho_{air} C_D A \tag{2.34}$$

In this case there is nothing to add or gain from a numerical analysis. Nevertheless, it is instructive to see how we could have dealt with this problem using numerical techniques. One way of solving the problem is, to write a computer code (e.g. in python) that computes from time instant to time instant the force on the jumper, and from that updates the velocity and subsequently the position.

```
some initial conditions
t = 0
x = x_0
v = 0
dt = 0.1

for i is 1 to N:
    compute F: formula
    compute new v: v[i+1] = v[i] - F[i]/m*dt
    compute new x: x[i+1] = x[i] + v[i]*dt
    compute new t: t[i+1] = t[i] + dt
```

You might already have some experience with numerical simulations. Figure 8 presents a script for the software Coach, which you might have encountered in secondary school.

```
'Stop condition is set
'Computations are based on Euler

x := x + flow_1*Δt1
v := v + flow_2*Δt1

t1 := t1 + Δt1

flow_1 := v
Fz := m*g
Fw := 6*d*d*v*v
f := Fz - Fw
a := f/m
flow_2 := a
```

```
t1 := 0            's
Δt1 := 0.01        's

x := 0             'm
v := 0             'm/s
m := 75            'kg
g := 9.81          'm/s^2
d := 2.5           'm

flow_1 := v        'm/s
Fz := m*g          'N
Fw := 6*d*d*v*v    'N
f := Fz - Fw       'N
a := f/m           'm/s^2
flow_2 := a        'm/s^2
```

Figure 2.8: An example of a numerical simulation made in Coach. At the left the iterative calculation process, at the right the initial conditions.

Example: The base jumper

Let us go back to the context of the base jumper and write some code.

First we take: $k = \frac{1}{2}\rho_{air}C_D A$ which eases writing. Newton's second law then becomes:

$$m\vec{a} = -m\vec{g} - k \mid v \mid \vec{v} \tag{2.35}$$

We rewrite this to a proper differential equation for v into a finite difference equation. That is, we go back to how we came to the differential equation:

$$m \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \vec{F}_{net} \tag{2.36}$$

with $\vec{F}_{net} = -m\vec{g} - k \mid v \mid \vec{v}$

On a computer, we cannot literally take the limit of Δt to zero, but we can make Δt very small. If we do that, we can rewrite the difference equation (thus not taken the limit):

$$\vec{v}(t+\Delta t) = \vec{v}(t) + \frac{\vec{F}}{m}\Delta t \tag{2.37}$$

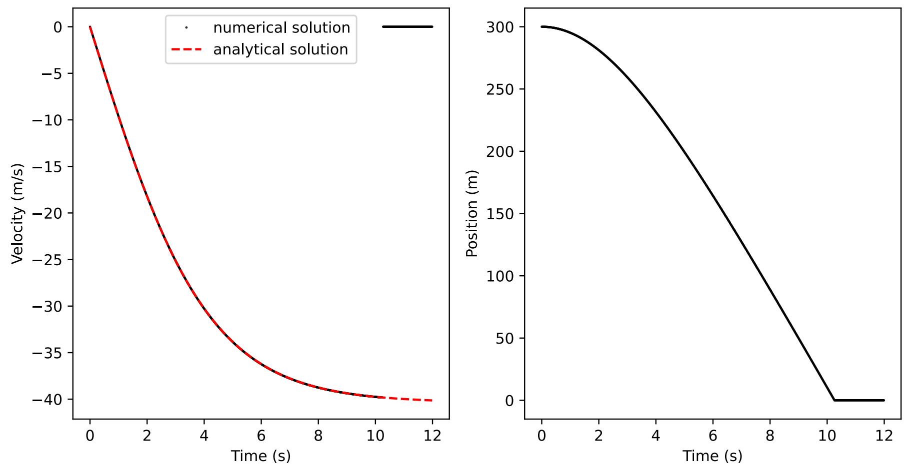
This expression forms the heart of our numerical approach. We will compute v at discrete moments in time: $t_i = 0, \Delta t, 2\Delta t, 3\Delta t, ...$ We will call these values v_i. Note that the force can be calculated at time t_i once we have v_i.

$$F_i = mg - k \mid v_i \mid v_i$$
$$v_{i+1} = v_i + \frac{F_i}{m}\Delta t \tag{2.38}$$

Similarly, we can keep track of the position:

$$\frac{dx}{dt} = v \Rightarrow x_{i+1} = x_i + v_i \Delta t \tag{2.39}$$

With the above rules, we can write an iterative code:



Important to note is the sign-convention which we adhere to. Rather than using v^2 we make use of $\mid v \mid v$ which takes into account that drag is always against the direction of movement. Note as well the similarity between the analytical solution and the numerical solution.

To come back to our initial problem:
It roughly takes 10 s to get close to terminal velocity (note that without friction the velocity would be 98 m/s). The building is not high enough to reach this velocity (safely).

Exercise 2.10: Base jumper with initial velocity 🌶️

Change the code so that the base jumper starts with an initial velocity along the z-direction.

Is the acceleration in the z-direction with and without initial velocity the same? Elaborate.

Exercise 2.12: Unit analysis 🌶️

Given the formula $F = kv^2$. Derive the unit of k, expressed only in SI-units

Exercise 2.13: Units based on physical constants[1] 🌶️ 🌶️

In physics, we assume that quantities like the speed of light (c) and Newton's gravitational constant (G) have the same value throughout the universe, and are therefore known as physical constants. A third such constant from quantum mechanics is Planck's constant ($\hbar$, h an with a bar). In high-energy physics, people deal with processes that occur at very small length scales, so our regular SI-units like meters and seconds are not very useful. Instead, we can combine the fundamental physical constants into different basis values.

1. Combine c, G and $\hbar$ into a quantity that has the dimensions of length.
2. Calculate the numerical value of this length in SI units (this is known as the Planck length).
3. Similarly, combine c, G and $\hbar$ into a quantity that has the dimensions of energy (indeed, known as the Planck energy) and calculate its numerical value.

2.1.6 Examples, exercises and experiments

Updated: 04 feb 2026

2.1.6.1 Worked examples

Warning

Here an example using the IDEA

2.1.6.2 Exercises

H5P exercise 1: Unit

```
### Your code
```

```
### Your code
```

2.1.6.3 Solutions

2.1.6.4 Experiments

Vectors

A hardboard with nails hammered at the sides. By attaching rubber bands between the nails, different vectors can be created. A spring scale can be used to measure the magnitude of the vector.Use three connected rubber bands and create a configuration like in . Measure the magnitude and direction of the force needed to pull the band loose from the nails. Make a drawing of the forces, both their magnitudes and directions and verify (by vector addition) that the net force is zero.

[1]Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Exercise 2.14: Reynolds numbers[2] 🌶️🌶️

Physicists often use *dimensionless quantities* to compare the magnitude of two physical quantities. Such numbers have two major advantages over quantities with numbers. First, as dimensionless quantities carry no units, it does not matter which unit system you use, you'll always get the same value. Second, by comparing quantities, the concepts 'big' and 'small' are well-defined, unlike for quantities with a dimension (for example, a distance may be small on human scales, but very big for a bacterium). Perhaps the best known example of a dimensionless quantity is the *Reynolds number* in fluid mechanics, which compares the relative magnitude of inertial and drag forces acting on a moving object:

$$\mathrm{Re} = \frac{\text{inertialforces}}{\text{dragforces}} = \frac{\rho v L}{\mu} \tag{2.40}$$

where ρ is the density of the fluid (either a liquid or a gas), v the speed of the object, L its size, and μ the viscosity of the fluid. Typical values of the viscosity are $1.0\ \mathrm{mPa} \cdot \mathrm{s}$ for water, $50\ \mathrm{mPa} \cdot \mathrm{s}$ for ketchup, and $1.8 \cdot 10^{-5}\ \mathrm{Pa} \cdot \mathrm{s}$ for air.

1. Estimate the typical Reynolds number for a duck when flying and when swimming (you may assume that the swimming happens entirely submerged). NB: This will require you looking up or making educated guesses about some properties of these birds in motion. In either case, is the inertial or the drag force dominant?
2. Estimate the typical Reynolds number for a swimming bacterium. Again indicate which force is dominant.
3. Oil tankers that want to make port in Rotterdam already put their engines in reverse halfway across the North sea. Explain why they have to do so.
4. Express the Reynolds number for the flow of water through a (circular) pipe as a function of the diameter D of the pipe, the volumetric flow rate (i.e., volume per second that flows through the pipe) Q, and the kinematic viscosity $\nu \equiv \eta / \rho$.
5. For low Reynolds number, fluids will typically exhibit so-called laminar flow, in which the fluid particles all follow paths that nicely align (this is the transparent flow of water from a tap at low flux). For higher Reynolds number, the flow becomes turbulent, with many eddies and vortices (the white-looking flow of water from the tap you observe when increasing the flow rate). The maximum Reynolds number for which the flow in a cylindrical pipe is typically laminar is experimentally measured to be about 2300. Estimate the flow velocity and volumetric flow rate of water from a tap with a 1.0 cm diameter in the case that the flow is just laminar.

Exercise 2.15: Powers of ten 🌶️

Calculate:

1. $10^{-4} \cdot 10^{-8} =$
2. $\frac{10^{6}}{10^{-19} \cdot 10^{4}} =$
3. $10^{12} \cdot 10^{-15} =$

[2]Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Exercise 2.16: Moving a box 🌶️

A box is on a frictionless incline of 10°. It is pushed upward with a force F_i for $\Delta t = 0.5$ s. It is then moving upward (inertia) but slows down due to gravity.

Below is a part of the python code. However, some essential elements of the code are missing indicated by **(..)**.

1. Include the correct code and run it.
2. Explain the two graphs, highlighting all essential features of the graph by relating these to the given problem.
3. At what time is the acceleration 0? At what time is the box back at its origin?

The above context is not very realistic as friction is neglected. We, however, can include friction easily as it is given by $\overrightarrow{F_w} = \mu \overrightarrow{F_N}$, with $\mu = 0.05$. Note that the direction of friction changes when the direction of the velocity changes!

4. Extend the code so that friction is included.

Exercise 2.17: Basejumper with parachute 🌶️ 🌶️

Our base jumper has yet not a soft landing. Luckily she has a working parachute. The parachute opens in 3.8 s reaching a total frontal area of 42.6 m^2. We can model the drag force using $\vec{F}_{drag} = k \mid v \mid \vec{v}$ with $k = 0.37$.

Write the code that simulates this jump of the base jumper with deploying the parachute. Show the (F_{drag}, t)-diagram and the (v, t)-diagram. What is the minimal height at which the parachute should be deployed?

Exercise 2.18: Circular motion 🌶️ 🌶️

Remember from secondary school circular motion, where the required force is given by $F = \frac{mv^2}{r}$. The corresponding vector form is: $\vec{F} = -\frac{mv^2}{r}\hat{r}$, or equivalent: $\vec{F} = -\frac{mv^2}{r^2}\vec{r}$. Now let's simulate that motion.

Assume:

- $m = 1$ kg
- $\vec{r}_0 = (3, 0)$ m
- $\vec{v}(0) = (0, 7)$ m/s

Write the code. You know the output already (a circle with radius of 3)!

Solution 2.19: Solution to Exercise 1

$$\begin{aligned} F &= kv^2 \\ &= [.]\left[\frac{\text{m}^2}{\text{s}^2}\right] \Rightarrow [.] = \left[\frac{\text{kg}}{\text{m}}\right] \end{aligned} \tag{2.41}$$

Solution 2.20: Solution to Exercise 2

The physical constants c, G and $\hbar$ have the following numerical values and SI-units:

$$\begin{aligned} c &= 2.99792458 \cdot 10^8 \text{ m/s} \\ G &= 6.674 \cdot 10^-11 \text{ m}^3/(\text{kg} \cdot \text{s}^2) \\ \hbar &= 1.054 \cdot 10^{-34} \text{ kgm}^2/\text{s} \end{aligned} \tag{2.42}$$

Note: the value of c is precise, i.e. by definition given this value. The second is defined via the frequency of radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom.

If we want to combine these three units into a length scale, $\mathcal{L}$, we try the following:

$$[\mathcal{L}] = [c]^A [G]^B [\quad \hbar]^C \tag{2.43}$$

What we mean here, is that the units of the quantities (denoted by [.]) left and right should be the same. Thus, we get:

$$m^1 = \left(\frac{m}{s}\right)^A \left(\frac{m^3}{kgs^2}\right)^B \left(\frac{kgm^2}{s}\right)^C \tag{2.44}$$

We try to find A, B, C such that the above equation is valid. We can write this equation as:

$$m^1 = m^{A+3B+2C} \cdot kg^{-B+C} \cdot s^{-A-2B-C} \tag{2.45}$$

If we split this into requirements for m, kg, s we get:

$$\begin{aligned} m &: 1 = A + 3B + 2C \\ kg &: 0 = C - B \\ s &: 0 = -A - 2B - C \end{aligned} \tag{2.46}$$

From the second equation we get $B = C$. Substitute this into the first and third and we find:

$$\begin{aligned} m &: 1 = A + 5B \\ s &: 0 = -A - 3B \end{aligned} \tag{2.47}$$

Add these two equations: $1 = 2B \rightarrow B = \frac{1}{2}$ and thus $C = \frac{1}{2}$ and $A = -\frac{3}{2}$.

So if we plug these values into our starting equation we see:

$$\mathcal{L} = \sqrt{\frac{\hbar G}{c^3}} = 1.62 \cdot 10^{-35} \text{ m} \tag{2.48}$$

We can repeat this for energy, $\mathcal{E}$:

$$[\mathcal{E}] = [c]^\alpha [G]^\beta [\quad \hbar]^\gamma \tag{2.49}$$

Note: the unit of energy, [J] needs to be written in terms of the basic units: $[J] = \text{kgm}^2/\text{s}^2$.

The outcome is: $\alpha = \frac{5}{2}$, $\beta = -\frac{1}{2}$, $\gamma = \frac{1}{2}$ and thus our energy is:

$$\mathcal{E} = \sqrt{\frac{\hbar c^5}{G}} = 1.96 \cdot 10^9 \text{ J} \tag{2.50}$$

Solution 2.21: Solution to Exercise 3

1. The size of a duck is on the order of 30 cm. It flies at a speed of about 70 km/h, that is 20 m/s. Thus we compute for the Reynolds number of a flying duck:

$$Re \equiv \frac{\rho v L}{\mu} = 4.0 \cdot 10^5 \tag{2.51}$$

Clearly, the inertial force is dominant.

What about a swimming duck? Now the velocity is much smaller: $v \approx 1\ \mathrm{m/s} = 3.6\ \mathrm{km/h}$. The viscosity of water is $\mu_w = 1.0\ \mathrm{mPa} \cdot \mathrm{s}$ and the water density is $1.0 \cdot 10^3\ \mathrm{kg/m^3}$. We, again, calculate the Reynolds number:

$$Re_w \equiv \frac{\rho v L}{\mu} = 3.0 \cdot 10^5 \tag{2.52}$$

Hence, also in this case inertial forces are dominant. This perhaps comes as a surprise, after all the velocity is much smaller and the viscosity much larger. However, the water density is also much larger!

2. For a swimming bacterium the numbers change. The size is now about 1μm and the velocity 60μ m/s (numbers taken from internet). That gives:

$$Re_b \equiv \frac{\rho v L}{\mu} = 6.0 \cdot 10^{-5} \tag{2.53}$$

and we see that here viscous forces are dominating.

3. For an oil tanker the Reynolds number is easily on the order of 10^8. Obviously, viscous forces don't do much. An oil tanker that wants to slow down cannot do so by just stopping the motors and let the drag force decelerate them: the Reynolds number shows that the viscous drag is negligible compared to the inertial forces. Thus, the tanker has to use ots engines to slow down. Again the inertia of the system is so large, that it will take a long time to slow down. And a long time, means a long trajectory.
4. For the flow of water through a (circular) pipe the Reynolds number uses as length scale the pipe diameter. We can relate the velocity of the water in the pipe tot the total volume that is flowing per second through a cress section of the pipe:

$$Q = \frac{\pi}{4} D^2 v \rightarrow v = \frac{4Q}{\pi D^2} \tag{2.54}$$

Thus we can also write Re as:

$$Re \equiv \frac{\rho v D}{\mu} = \frac{4Q}{\pi \frac{mu}{\rho} D^2} = \frac{4Q}{\pi \nu D^2} \tag{2.55}$$

1. If $Re = 2300$ for the pipe flow, we have:

$$Re = \frac{vD}{\nu} = 2300 \rightarrow v = \frac{2300\nu}{D} \tag{2.56}$$

with $\nu = 1.0 \cdot 10^{-6}\ \mathrm{m^2/s}$ and $D = 1.0 \cdot 10^{-2}$ m we find: $v = 0.23$ m/s and $Q = 1.8 \cdot 10^{-5}\ \mathrm{m^3/s} = 0.018$ liter/s.

Solution 2.22: Solution to Exercise 4

1. $= 10^{-12}$
2. $= 10^{21}$
3. $= 10^{-3}$

Solution 2.23: Solution to Exercise 5

```
# Moving a box

## Importing libraries
import numpy as np
import matplotlib.pyplot as plt


part_4 = 1 # Turn to 0 for first part

## Constants
m = 2 #kg
F = 30 #N
g = 9.81 #m/s^2
theta = np.deg2rad(10) #degrees

mu = 0.02
F_N = m*g*np.cos(theta) #N

## Time step
dt = 0.01 #s
t = np.arange(0, 10, dt) #s
t_F_stop = 0.5

## Initial conditions
x = np.zeros(len(t)) #m
v = np.zeros(len(t)) #m/s


## Loop to calculate position and velocity
for i in range(0, len(t)-1):
    if t[i] < t_F_stop:
        a = F/m - g*np.sin(theta) - F_N*mu*np.where(v[i] != 0,
np.sign(v[i]), 0)*part_4
    else:
        a = -g*np.sin(theta) - F_N*mu*np.where(v[i] != 0, np.sign(v[i]),
0)*part_4
    v[i+1] = v[i] + a*dt
    x[i+1] = x[i] + v[i]*dt

## Plotting results
figs, axs = plt.subplots(1, 2, figsize=(10, 5))

axs[0].set_xlabel('Time (s)')
axs[0].set_ylabel('Velocity (m/s)')
axs[0].plot(t, v, 'k.', markersize=1)


axs[1].set_xlabel('Time (s)')
axs[1].set_ylabel('Position (m)')
axs[1].plot(t, x, 'k.', markersize=1)



plt.show()
```

Solution 2.24: Solution to Exercise 6

```
# Simulation of a base jumper

## Importing libraries
import numpy as np
import matplotlib.pyplot as plt

## Constants
A = 0.7 #m^2
m = 75 #kg
k = 0.37 #kg/m
g = 9.81 #m/s^2

## Time step
dt = 0.01 #s
t = np.arange(0, 12, dt) #s

## Initial conditions
z = np.zeros(len(t)) #m
v = np.zeros(len(t)) #m/s
z[0] = 300 #m

## Deploy parachute
A_max = 42.6 #m^2
t_deploy_start = 2 #s
dt_deploy = 3.8 #s

## Loop to calculate position and velocity
for i in range(0, len(t)-1):
    F = - m*g - k*A*abs(v[i])*v[i]  #N
    v[i+1] = v[i] + F/m*dt #m/s
    z[i+1] = z[i] + v[i]*dt #m
    # Check if the jumper is on the ground
    if z[i+1] < 0:
        break
    # Deploy parachute
    if t[i] > t_deploy_start and t[i] < t_deploy_start + dt_deploy:
        A += (A_max - A)/dt_deploy*dt

## Plotting results
figs, axs = plt.subplots(1, 2, figsize=(10, 5))

axs[0].set_xlabel('Time (s)')
axs[0].set_ylabel('Velocity (m/s)')

axs[0].plot(t, v, 'k.', markersize=1, label='numerical solution')
axs[0].vlines(t_deploy_start, v[t==t_deploy_start],0, color='gray',
linestyle='--', label='parachute deploy')

axs[0].legend()

axs[1].set_xlabel('Time (s)')
axs[1].set_ylabel('Position (m)')

axs[1].plot(t, z, 'k.', markersize=1)
axs[1].vlines(t_deploy_start, 150,300, color='gray', linestyle='--',
label='parachute deploy')


plt.show()
```

Solution 2.25: Solution to Exercise 7

```
import numpy as np
import matplotlib.pyplot as plt

F = 49/3
m1 = 1
dt = 0.001
t = np.arange(0, 100, dt) # s

x1 = np.zeros(len(t)) # m
x1[0] = 3
y1 = np.zeros(len(t)) # m
vx = 0
vy = 7


for i in range(0, len(t)-1):
    ax = -F*(x1[i]-0)/np.sqrt(x1[i]**2 + y1[i]**2)/m1
    ay = -F*(y1[i]-0)/np.sqrt(x1[i]**2 + y1[i]**2)/m1
    vx = vx + ax*dt
    vy = vy + ay*dt
    x1[i+1] = x1[i] + vx*dt
    y1[i+1] = y1[i] + vy*dt


plt.figure(figsize=(4,4))
plt.plot(x1, y1, 'k.', markersize=1)
plt.xlabel('x (m)')
plt.ylabel('y (m)')
plt.show()
```

2.2 Newton's Laws

Updated: 04 feb 2026 Now we turn to one of the most profound breakthroughs in the history of science: the laws of motion formulated by Isaac Newton . These laws provide a systematic framework for understanding how and why objects move. They form the backbone of classical mechanics. Using these three laws we can predict the motion of a falling apple, a car accelerating down the road, or a satellite orbiting Earth (though some adjustments are required in this context to make use of e.g. GPS!). More than just equations, they express deep principles about the nature of force, mass, and interaction.

In this chapter, you will begin to develop the core physicist's skill: building a simplified model of the real world, applying physical principles, and using mathematical tools to reach meaningful conclusions.

2.2.1 Newton's Three Laws

Much of physics, in particular Classical Mechanics, rests on three laws that carry Newton's name:

Newton's first law (N1)

Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare.

If no force acts on an object, the object moves with constant velocity.

Newton's second law (N2)

Mutationem motus proportionalem esse vi motrici impressæ, & fieri secundum lineam rectam qua vis illa imprimitur.

If a (net) force acts on an object, the momentum of the object will change according to:

$$\frac{d\vec{p}}{dt} = \vec{F} \tag{2.57}$$

Newton's third law (N3)

Actioni contrariam semper & æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales & in partes contrarias dirigi.

If object 1 exerts a force $\vec{F}_{12}$ on object 2, then object 2 exerts a force $\vec{F}_{21}$ equal in magnitude and opposite in direction on object 1:

$$\vec{F}_{21} = -\vec{F}_{12} \tag{2.58}$$

N1 has, in fact, been formulated by Galileo Galilei. Newton has, in his N2, build upon it: N1 is included in N2, after all:
if $\vec{F} = 0$, then $\frac{d\vec{p}}{dt} = 0 \rightarrow \vec{p} = constant \rightarrow \vec{v} = constant$, provided m is a constant.

Most people know N2 as

$$\vec{F} = m\vec{a} \tag{2.59}$$

For particles of constant mass, the two are equivalent:
if $m = constant$, then

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a} \tag{2.60}$$

Nevertheless, in many cases using the momentum representation is beneficial. The reason is that momentum is one of the key quantities in physics. This is due to the underlying conservation law that we will derive in a minute. Momentum is a more fundamental concept in physics than acceleration. That is another reason why physicists prefer the second way of looking at forces.

Moreover, using momentum allows for a new interpretation of force: force is that quantity that - provided it is allowed to act for some time interval on an object - changes the momentum of that object. This can be formally written as:

$$d\vec{p} = \vec{F}dt \leftrightarrow \Delta\vec{p} = \int \vec{F}dt \tag{2.61}$$

The latter quantity $\vec{I} \equiv \int \vec{F}dt$ is called the impulse.

Note

Momentum is in Dutch **impuls**; the English **impulse** is in Dutch **stoot**.

In Newton's laws, velocity, acceleration and momentum are key quantities. We repeat here their formal definition.

Definition

$$\begin{aligned} \text{velocity} : \vec{v} \equiv \lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} &= \frac{d\vec{r}}{dt} \\ \text{acceleration} : \vec{a} \equiv \lim_{\Delta t\to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} &= \frac{d\vec{v}}{dt} \\ \text{momentum} : \vec{p} \equiv m\vec{v} = m\frac{d\vec{r}}{dt} \end{aligned} \tag{2.62}$$

2.2.2 Conservation of Momentum

From Newton's 2^{nd} and $3^{r}d$ law we can easily derive the law of conservation of momentum. Assume there are only two point-particle (i.e. particles with no size but with mass), that exert a force on each other. No other forces are present. From N2 we have:

$$\begin{aligned} \frac{d\vec{p}_1}{dt} &= \vec{F}_{21} \\ \frac{d\vec{p}_2}{dt} &= \vec{F}_{12} \end{aligned} \tag{2.66}$$

From N3 we know:

$$\vec{F}_{21} = -\vec{F}_{12} \tag{2.67}$$

And, thus by adding the two momentum equations we get:

$$\left.\begin{aligned} \tfrac{d\vec{p}_1}{dt} &= \vec{F}_{21} \\ \tfrac{d\vec{p}_2}{dt} &= \vec{F}_{12} = -\vec{F}_{21} \end{aligned}\right\} \Rightarrow \tag{2.68}$$

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0 \rightarrow \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0 \tag{2.69}$$

Exercise 2.26: 🌶️

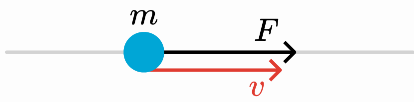
Consider a point particle of mass m, moving at a velocity v_0 along the x-axis. At $t = 0$ a constant force acts on the particle in the positive x-direction. The force lasts for a small time interval Δt.

What is the velocity of the particle for $t > \Delta t$?

Solution 2.27: Solution to Exercise 1

Interpret

First we make a sketch.



This is obviously a 1-dimensional problem. So, we can leave out the vector character of e.g. the force.

Develop

We will use $dp = Fdt$:

$$dp = Fdt \Rightarrow \Delta p = \int_0^{\Delta t} Fdt = F\Delta t \rightarrow \tag{2.63}$$

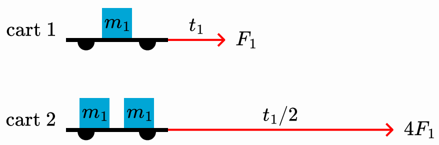
$$p(\Delta t) = p(0) + F\Delta t = mv_0 + F\Delta t \rightarrow \tag{2.64}$$

$$v(\Delta t) = v_0 + \frac{F}{m}\Delta t \tag{2.65}$$

Note that this example could also be solved by N2 in the form of $F = ma$. It is merely a matter of taste.

$$\Rightarrow \vec{p}_1 + \vec{p}_2 = const\ i.e. does not depend on time \tag{2.70}$$

Exercise 2.29: A pushing contest 🌶️



Exercise 2.31: Newton's third law 🌶️

The base jumper from chapter 1 just jumped from the tall building. According to Newton's third law there are two coupled forces. Which are these, and what is the consequence of these two forces?

Solution 2.32: Solution to Exercise 3

The gravitational force acts from the earth on the jumper. Newton's law states that the jumper thus acts a gravitational force on the earth. Hence, the earth accelerates towards the jumper!

Although this sounds silly, when comparing this idea to the sun and the planets, we must draw the conclusion that the sun is actually wobbling as it is pulled towards the various planets! See also this animated explanation

Note the importance of the last conclusion: **if objects interact via a mutual force then the total momentum of the objects cannot change.** No matter what the interaction is. This notion is easily extended to more interacting particles. The crux is that particles interact with one another via forces that obey N3. Thus for three interacting point particles we would have (with $\vec{F}_{ij}$ the force from particle *i* felt by particle *j*):

$$\begin{aligned} \tfrac{d\vec{p}_1}{dt} &= \vec{F}_{21} + \vec{F}_{31} \\ \tfrac{d\vec{p}_2}{dt} &= \vec{F}_{12} + \vec{F}_{32} = -\vec{F}_{21} + \vec{F}_{32}\} \\ \tfrac{d\vec{p}_3}{dt} &= \vec{F}_{13} + \vec{F}_{23} = -\vec{F}_{31} - \vec{F}_{32} \end{aligned} \tag{2.71}$$

Sum these three equations:

$$\begin{gathered} \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} + \frac{d\vec{p}_3}{dt} = 0 \rightarrow \frac{d}{dt}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) = 0 \\ \Rightarrow \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = const.\ i.e. doesnotdependontime \end{gathered} \tag{2.72}$$

For a system of N particles, extension is straight forward.

Intermezzo: Isaac Newton

At the end of the year of Galilei's death, Isaac Newton was born in Woolsthorpe-by-Colsterworth in England. He is regarded as the founder of classical mechanics and with that he can be seen as the father of physics.Isaac Newton (1642-1727). From Wikimedia Commons, public domain.In 1661, he started studying at Trinity College, Cambridge. In 1665, the university temporarily closed due to an outbreak of the plague. Newton returned to his home and started working on some of his breakthroughs in calculus, optics and gravitation. Newton's list of discoveries is unsurpassed. He 'invented' calculus (at about the same time and independent of Leibniz). Newton is known for 'the binomium of Newton', the cooling law of Newton. He proposed that light is made of particles and formulated his law of gravity. Finally, he postulated his three laws that started classical mechanics and worked on several ideas towards energy and work. Much of our concepts in physics are based on the early ideas and their subsequent development in classical mechanics. The laws and rules apply to virtually all daily life physical phenomena and only require adaptation when we go to very small scale or extreme velocities and cosmology. In what follows, we will follow his footsteps, but in a modern way that we owe to many physicist and mathematicians that over the years shaped the theory of classical mechanics in a much more comprehensive form. We do not only stand on shoulders of giants, we stand on a platform carried by many.Interesting to know is that his mentioning of standing on shoulders can be interpreted as a sneer towards Robert Hooke (1635-1703), with he was in a fight with over several things. Hooke was a rather short man… See, e.g., .

Important

In Newtonian mechanics time does not have a preferential direction. That means, in the equations derived based on the three laws of Newton, we can replace t by $-t$ and the motion will have different sign, but that's it. The path/orbit will be the same, but traversed in opposite direction. Also in special relativity this stays the same.

However, in daily life we experience a clear distinction between past, present and future. This difference is not present in this lecture at all. Only by the second of law thermodynamics the time axis obtains a direction, more about this in classes on Statistical Mechanics.

2.2.3 Newton's laws applied

2.2.3.1 Force addition, subtraction and decomposition

Newton's laws describe how forces affect motion. Applying them often requires combining multiple forces acting on an object , see Figure 4. This is done through vector addition, subtraction, and decomposition—allowing us to find the net force and to analyze its components in different directions (see this chapter in the book on linear algebra for a full elaboration on vector addition and subtraction).

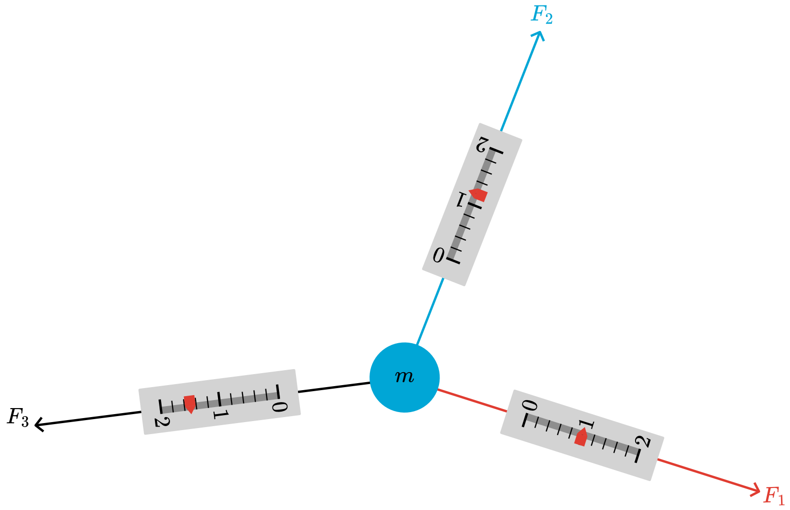


Figure 2.33: Three forces acting on a particle. In which direction will it accelerate?

Example: Three forces acting on a particle

Consider three forces acting on a particle:

$\vec{F}_1 = \binom{1}{0}$, $\vec{F}_2 = \binom{1}{1}$ and $\vec{F}_3 = \binom{-1}{-0.5}$

What is the net force acting on the particle and in which direction will the particle accelerate?

Example: Incline

The box in Figure 5 is at rest. Calculate the frictional force acting on the box.

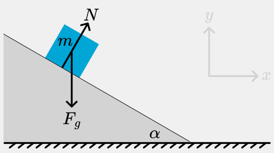


Figure 2.36: A box is at rest on an incline.

Develop

Exercise 2.34: Forces acting on a particle in 3D

Three forces act on a particle with mass m:

$$\vec{F}_1 = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}, \vec{F}_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} and \vec{F}_3 = \begin{pmatrix} -1 \\ -0.5 \\ 1 \end{pmatrix} \tag{2.73}$$

Determine the acceleration of this particle.

Solution 2.35: Solution to Exercise 4

$$\begin{aligned}\vec{F}_{net} &= \sum \vec{F}_i = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\ &= \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ -0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1-1 \\ 0+1+-0.5 \\ -4+3+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}\end{aligned} \tag{2.74}$$

Hence, the net force acting on the particle is $\sqrt{1^2 + .5^2} = 1.1N$ and the particle will accelerate in the direction $\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$, in essence just like in the previous example. The magnitude of the acceleration is $a = F/m$ and can only be calculated when the mass of the particle is specified.

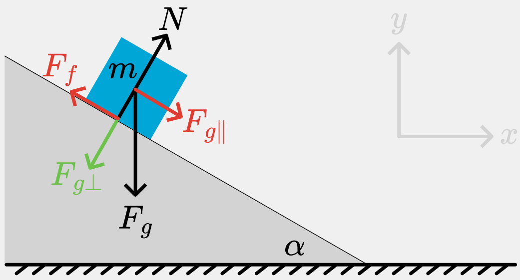
As the box is not moving (i.e. it has a constant velocity) the sum of forces on the box must be equal to zero. In the sketch we see two forces that clearly do not add up to zero. A third force is needed.

Evaluate

If we assume that only friction as a third force is present, we require:

$$\sum_i \vec{F}_i = 0 \Rightarrow \vec{F}_g + \overrightarrow{F_N} + \vec{F}_f = 0 \Rightarrow \vec{F}_f = -\vec{F}_g - \vec{F}_N \tag{2.75}$$

We can progress further by assuming that the friction force acts parallel to the slope. With this assumption, we can decompose gravity in its components perpendicular to the slope and parallel to the slope.



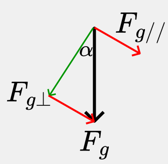
$$\vec{F}_g = \vec{F}_{g//} + \vec{F}_{g\perp} \tag{2.76}$$

The normal force exactly balances the perpendicular component: that is what a normal force does. Friction balances the parallel component of gravity:

$$\vec{F}_f + \vec{F}_{g//} = 0 \rightarrow \vec{F}_f = -\vec{F}_{g//} \tag{2.77}$$

and its magnitude is thus $F_f = F_g \sin\alpha$

Reminder



> Remember from secondary school how to break down a force vector into components.

2.2.3.2 Acceleration due to gravity

In most cases the forces acting on an object are not constant. However, there is a classical case that is treated in physics (already at secondary school level) where only one, constant force acts and other forces are neglected. Hence, according to Newton's second law, the acceleration is constant.

When we first consider only the motion in the z-direction, we can derive:

$$a = \frac{F}{m} = const. \tag{2.78}$$

Hence, for the velocity:

$$v(t) = v_0 + \int_{t_0}^{t_e} a dt = a(t_e - t_0) + v_0 \tag{2.79}$$

assuming $t_0 = 0 and t_e = t \Rightarrow v(t) = v_0 + at$ the position is described by

$$s(t) = \int_0^t v(t) dt = \int_0^t at + v_0 dt = \frac{1}{2}at^2 + v_0 t + s_0 \tag{2.80}$$

Rearranging:

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0 \tag{2.81}$$

Example: 2D-motion

We only considered motion in the vertical direction, however, objects tend to move in three dimension. We consider now the two-dimensional situation, starting with an object which is horizontally thrown from a height.

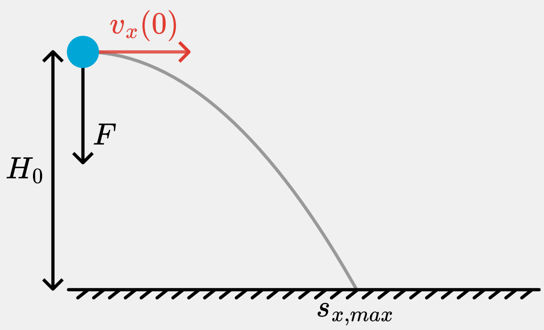


Figure 2.42: A sketch of the situation where an object is thrown horizontally and the horizontal distance should be calculated.

Exercise 2.39: Tossing a stone in the air 🌶️

At a height of 1.5 m a stone is tossed in the air with a velocity of 10 m/s.

1. Calculate the maximum height that it reaches.
2. Calculate the time it takes to reach this point.
3. Calculate with which velocity it hits the ground.

Solution 2.40: Solution to Exercise 5

Interpret

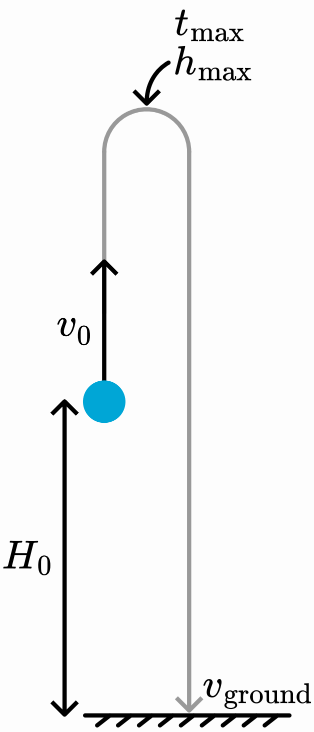


Figure 2.41: A free body diagram of the situation with all relevant quantities.

Only gravity acts on the stone (in the downward direction). We will call the position of the stone at time t: $s(t)$
Initial conditions: $t = 0 \rightarrow s(0) = s_0 = 1.5\ m and \dot{s} = v = v_0 = 10\ \text{m/s}$

Develop

1. $s(t) = \frac{1}{2}at^2 + v_0 t + s_0$ Highest point reached when $\dot{s} = 0$
2. $\Delta t = \frac{\Delta v}{a}$
3. $s(t) = \frac{1}{2}at^2 + v_0 t + s_0$. We are interested in the stone hitting the ground. Thus, solve for $s(t) = 0$ to find at what time this happens.

Evaluate

1. $\dot{s} = at + v_0 = -gt + v_0 = 0 \Rightarrow t = 1.02s$

$s(1.02) = -\frac{1}{2} * 9.81 * 1.02^2 + 10 * 1.02 + 1.5 = 6.6m$

1. See above.
2. $s(t) = \frac{1}{2}at^2 + v_0 t + s_0 = s_e$

$t = \frac{-v_0 \pm \sqrt{v_0^2 - 4(\frac{1}{2}a(s_0 - s_e))}}{2\frac{1}{2}a} = \frac{-10 \pm \sqrt{10^2 - 4(\frac{1}{2}(-9.81)(1.5))}}{-9.81} = 2.18s$
$v(2.18) = \dot{s}(2.18) = v_0 + at = 10 - 9.81 * 2.18 = -11.3\ \text{m/s}$
Note that $t = -0.14s$ is another solution, but not physically realistic.

Assess

The times we calculated are in the right order: First stone is tossed (at $t_0 = 0$), then it reaches its highest point (at $t_m = 1.02$ s). After that it falls and hits the ground at $t_e = 2.18$ s. Thus $t_0 < t_m < t_e$.

Furthermore, the velocity upon impact with the earth is negative as it should: the stone is falling downward. Its magnitude is on the order of the initial upward velocity, which makes sense. Finally, our answers have the right units.

NOTE: Some of these solutions can be derived more easily using the concept of *conservation of energy* which will be covered in one of the next chapters.

In the situation given in Figure 9 the object is thrown with a horizontal velocity of v_{x0}. As no forces in the horizontal direction act on the object (N1), its horizontal motion can be described by

$$s_x(t) = v_{x0}t \tag{2.82}$$

In the vertical direction only the gravitational force acts (N2), hence the motion can be described by (26). Taking the y-direction upward, a starting height $y(0) = H_0$ and $v_y(0) = 0$ it becomes:

$$s_y(t) = H_0 - \frac{1}{2}gt^2 \tag{2.83}$$

The total horizontal traveled distance of the object before hitting the ground then becomes:

$$s_{x,max} = v_x\sqrt{\frac{2H_0}{g}} \tag{2.84}$$

This motion is visualized in Figure 10. The trajectory is shown with s_x on the horizontal axis and s_y on the vertical axis. At regular time intervals Δt, velocity vectors are drawn to illustrate the motion. Note that the horizontal and vertical components of velocity, v_x and v_y, vary independently throughout the trajectory. Moreover, $\vec{v}(t)$ is the tangent of $s(t)$.

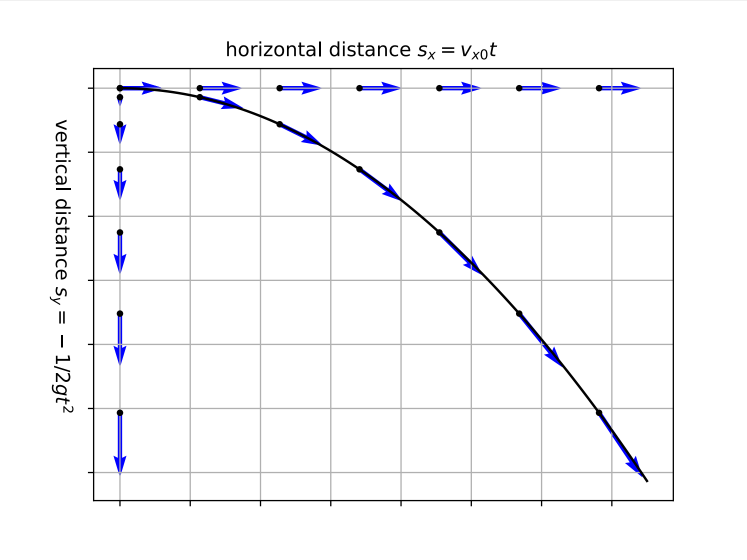


Figure 2.43: The parabolic motion is visualized with blue velocity vectors $v, v_x andv_y$ shown at various points along the trajectory.

Danger

Understand that the case above is specific in physics: in most realistic contexts multiple forces are acting upon the object. Hence the equation of motion does not become $s(t) = s_0 + v_0t + 1/2at^2$

Exercise 2.44: Horizontal throw 🌶️

Derive the above expression (29) yourselves.

Exercise 2.45: Projectile motion 🌶 🌶

Watch the recording below. What happens with the horizontal distance traveled per time unit? And with the vertical distance traveled?

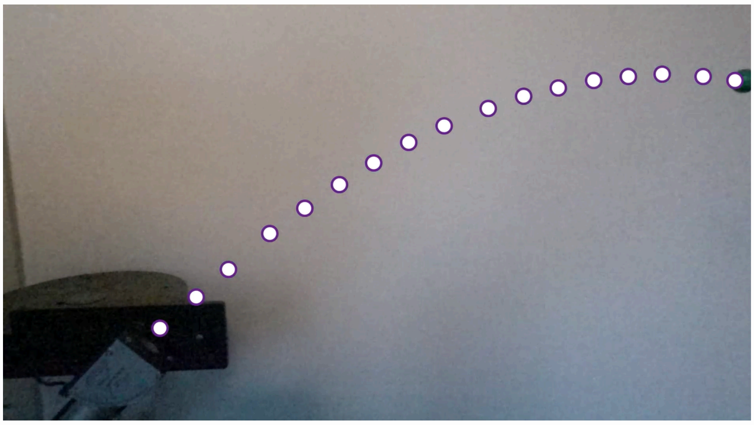


Figure 2.46: A parabolic motion visualized, with the position stored per time unit :alt: A short video of a small ball being shot upward at an angle. For each frame, it position is marked by a dot. The dots make up a parabola.

Assume the object with mass m_1 is shot from the ground with a velocity of v_0 at an angle of θ. Derive where the object hits the ground in terms of m_1, v_0 and θ.

How does the distance traveled changes when the mass of the object is doubled $m_2 = 2m_1$?

2.2.3.3 Frictional forces

There are two main types of frictional force:

- **Static friction** prevents an object from starting to move. It adjusts in magnitude up to a maximum value, depending on how much force is trying to move the object. This maximum is given by

$$F_{static,max} = \mu_s N \tag{2.100}$$

where μ_s is the coefficient of static friction and N is the normal force. If the applied force exceeds this maximum, the object begins to slide.

- **Kinetic (dynamic) friction** opposes motion once the object is sliding. Its magnitude is generally constant and given by

$$F_{kinetic} = \mu_k N \tag{2.101}$$

where μ_k is the coefficient of kinetic friction. This force does not depend on the velocity of the object, only on the normal force and surface characteristics.

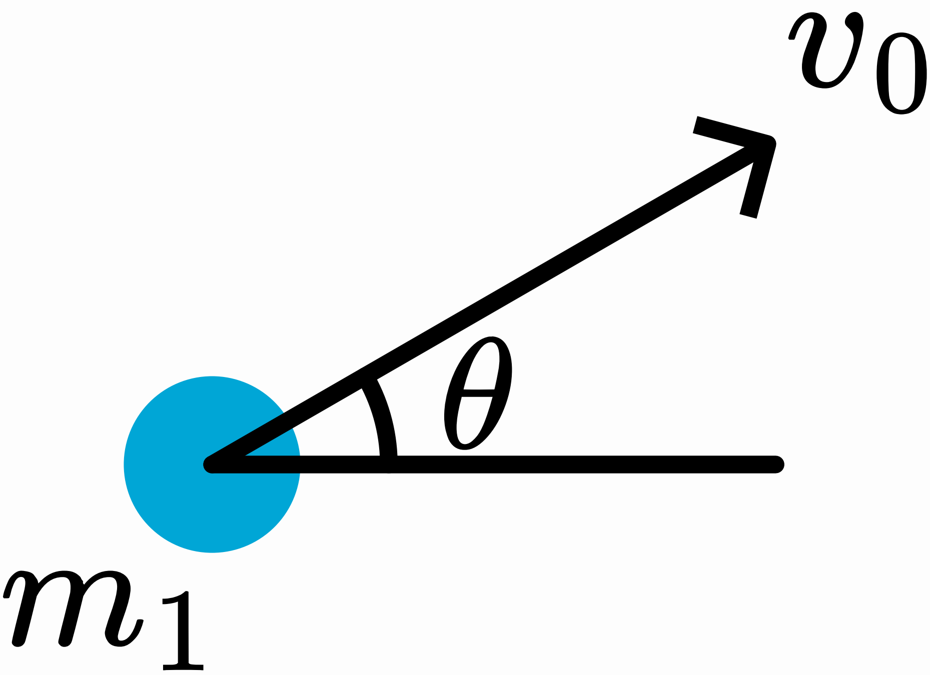
Friction always acts opposite to the direction of intended or actual motion and is essential in both preventing and controlling movement.

Material Pair	Static Friction (μ_s)	Kinetic Friction (μ_k)
Rubber on dry concrete	1.0	0.8
Steel on steel (dry)	0.74	0.57
Wood on wood (dry)	0.5	0.3
Aluminum on steel	0.61	0.47
Ice on ice	0.1	0.03

Solution 2.47: Solution to Exercise 7

The horizontal traveled distance is the same per time unit. For the vertical traveled distance it decreases until $v_y = 0$ and then increases.

Interpret



Develop

The basic formulas are:

$$s_x(t) = v_x t \tag{2.85}$$

and

$$s_y(t) = v_y t - 1/2gt^2 \tag{2.86}$$

Evaluate

The horizontal traveled distance is given by:

$$s_x(t) = v_x t = v_0 \cos(\theta) t \tag{2.87}$$

The time the object stays in the air is

$$s_y(t) = v_y t - 1/2gt^2 = 0 \Rightarrow t = 0t = \frac{2v_y}{g} = \frac{2v_0 \sin(\theta)}{g} \tag{2.88}$$

Hence, the maximum distance traveled is:

$$s_x(t) = v_x t = v_0 \cos(\theta) \frac{2v_0 \sin(\theta)}{g} = \frac{2v_o^2 \sin(\theta) \cos(\theta)}{g} \tag{2.89}$$

Note that the distance traveled is independent of the mass!

Glass on glass	0.94	0.4

Exercise 2.49: Constant acceleration due to gravity 🌶

We assumed a constant acceleration due to gravity. However, the gravitational force is given by $F = -G\frac{mM}{r^2}$.
Calculate at what height above the earth the acceleration due to gravity has 'significantly' changed from 9.81m/s^2, say to 9.80m/s^2.

Solution 2.50: Solution to Exercise 8

The acceleration of gravity is found by setting the gravitation force equal to $-mg$:

$$-G\frac{mM}{r^2} = -mg(r) \Rightarrow g(r) = G\frac{M}{r^2} \tag{2.90}$$

with M the mass of the earth.

At the surface of the earth, $r = R_e$ we have for the value of $g_e = 9.81$ m/s^2. We look for the height above the earth surface where g has dropped to 9.80 m/s^2. If we call this height H, we write for the distance to the center of the earth $r = R_e + H$.

Thus, we look for $\frac{g(r)}{g_e} = \frac{9.80(\text{m/s}^2)}{9.81(\text{m/s}^2)} = 0.999$:

$$\frac{g(r)}{g_e} = \frac{GM/r^2}{GM/R_e^2} \rightarrow \frac{R_e^2}{r^2} = \frac{R_e^2}{(R_e + H)^2} = \frac{9.80}{9.81} = 0.999 \tag{2.91}$$

If we solve H from this equation we find: $H = 3.25$ km (we used $R_e = 6378$ km).

Note

We could have also looked at the ratios (between g and r), and found that $R_2 = \sqrt{.999} \cdot 6378 = 6374.8$ km. Hence, $H = 3.2$ km.

If we would have said: 'significant change' in means $g \rightarrow 9.81 \rightarrow 9.71$ m/s^2, we would have found $H = 32.8$ km.

Copper on steel	0.53	0.36
Teflon on Teflon	0.04	0.04
Rubber on wet concrete	0.6	0.5

Exercise 2.51: A rocket in space 🌶

A rocket moves freely horizontal through space. At position $x = 2$ it turns on its propulsion. At position $x = 4$ it turns off its propulsion. The force due to this propulsion is directed perpendicular to the x-direction.

Provide a sketch of its movement highlighting all important parts.

Exercise 2.52: Particle movement 🌶 🌶

Consider a particle which will travel a distance x. Find two different mathematical expressions for a force acting on the particle in such a way that the particle will travel the same distance in the same time for each $F(t)$ compared to a particle which travels at constant speed. Assume no initial velocity for the two particles.

Solution 2.53: Solution to Exercise 10

Uniform motion ($F = ma = 0 \to s = v_0 t$).

Constant acceleration $a = const \to s = 1/2at^2$, with $a = \frac{2v_0^2}{s}$.

Consider the third being a harmonic oscillating force field: $F(t) = A\sin(2\pi f t)$ Then the equation of motion becomes:

$$a = F/m = \frac{A}{m}\sin(2\pi f t) \tag{2.92}$$

$$v = \int a dt = \frac{A}{m 2\pi f}\cos(2\pi f t) + C_0 \tag{2.93}$$

Assuming $v(0) = 0 \to C_0 = -\frac{A}{m2\pi f}$

And,

$$x = \int v dt = \frac{A}{m(2\pi f)^2}\sin(2\pi f t) + C_0 t + C_1 \tag{2.94}$$

Assuming $x(0) = 0 \to C_1 = 0$

Hence:

$$x = \frac{A}{m(2\pi f)^2}\sin(2\pi f t) - \frac{A}{m2\pi f}t \tag{2.95}$$

Now, finding traveling the same distance in the same time AND the harmonic oscillation is complete (hence, $f = \frac{1}{t_e}$):

$$v_0 t_e = \frac{A t_e^2}{m(2\pi)^2}\sin(2\pi) - \frac{A t_e}{m 2\pi}t_e \tag{2.96}$$

$$v_0 t_e = -\frac{A t_e^2}{m 2\pi} \tag{2.97}$$

$$v_0 = -\frac{A t_e}{m 2\pi} \tag{2.98}$$

$$\frac{m}{A} = -\frac{t_e}{v_0 2\pi} \tag{2.99}$$

Leather on wood	0.56	0.4

Values are approximate and can vary depending on surface conditions.

Note

Not always are the friction coefficients constants. They may, for instance, depend on the relative velocity between the two materials.

2.2.3.4 Momentum example

The above theoretical concept is simple in its ideas:

- a particle changes its momentum whenever a force acts on it;
- momentum is conserved;
- action = - reaction.

Exercise 2.54: Block on an incline

A block with mass m is put on an inclined plane of which we can change the inclination angle θ.

1. Determine the angle at which it starts to slide in terms of mass m, inclination angle θ, acceleration due to gravity g and coefficient of static friction μ_s.
2. Once it starts to slide, it will accelerate. Determine its acceleration in terms of mass m, inclination angle θ, acceleration due to gravity g and coefficient of kinetic friction μ_f.

Solution 2.55: Solution to Exercise 11

1. There a two forces acting on m parallel to the inclined plane: friction and gravity's component parallel to the slope. These two determine the motion along the slope: if we tilt the plane the component of gravity parallel to the slope gets bigger. The particle will start moving when we pass: $F_{g_x} = F_s \rightarrow mg\sin(\theta) = mg\mu_s\cos(\theta) \Rightarrow \theta_{max} = \tan^{-1}(\mu_s)$
2. Once the particle is sliding downward, gravity and the kinetic friction determine how fast:

$$F_{net} = F_{g_x} - F_f \rightarrow ma = mg\sin(\theta) - mg\mu_k\cos(\theta) \Rightarrow \tag{2.102}$$

and

$$a = g(\sin(\theta) - \mu_k\cos(\theta)) \tag{2.103}$$

But it is incredible powerful and so generic, that finding when and how to use it is much less straight forward. The beauty of physics is its relatively small set of fundamental laws. The difficulty of physics is these laws can be applied to almost anything. The trick is how to do that, how to start and get the machinery running. That can be very hard. Luckily there is a recipe to master it: it is called practice.

2.2.4 Forces & Inertia

Newton's laws introduce the concept of force. Forces have distinct features:

- forces are vectors, that is, they have magnitude and direction;
- forces change the motion of an object:

Exercise 2.56: Frictional force[3]

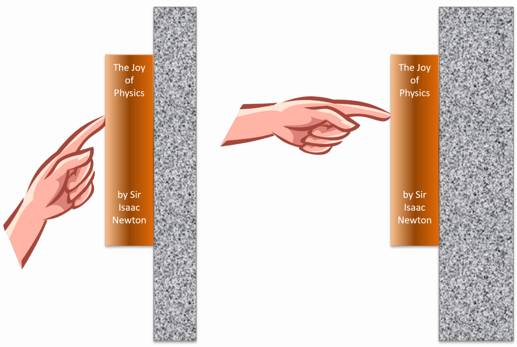


Figure 2.57: A book is held in place against a wall. The magnitude of the force applied by the hand on the book is the same in the left and the right scenarios.

[3]Kortemeyer (2025)

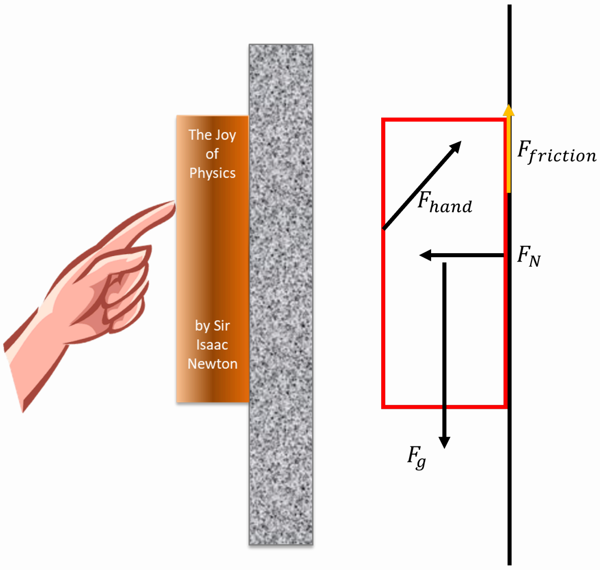
Solution 2.58: Solution to Exercise 12

Interpret

The question resolves around which forces are acting on the book. What we already know is that the book is not moving in any direction and thus the net force must be zero.

Develop

Let's draw the free body diagram of the book for the left scenario:



We see two forces with a horizontal component: the force of the hand and the normal force of the wall. The forces with a vertical component are the gravitational force, the force of the hand and the frictional force.

Evaluate

From N1 and N3 we know that the net force must be zero. Looking at Figure 14, we can write for the horizontal and vertical directions:

$$\Sigma F_x = F_{hand,x} - F_{normal} = 0 \tag{2.104}$$

$$\Sigma F_y = F_{hand,y} + F_{friction} - F_{gravity} = 0 \tag{2.105}$$

Now it is clear that if the force of the hand is perpendicular to the wall, the frictional force must compensate for the gravitational force. If the hand applies a force at an angle, the normal force decreases (as the horizontal component of the hand's force decreases). As the force of the hand has a verticale component, the frictional force of the wall can decrease as well. **Note** that the force of the hand might be so big that the direction of the frictional force flip!

Assess

We can check that the direction of the force of our hand matters by doing the experiment Exercise 12.

- they change the velocity, i.e. they accelerate the object

Exercise 2.60: 🌶️

A point particle (mass m) is dropped from rest at a height h above the ground. Only gravity acts on the particle with a constant acceleration g ($= 9.813\ \mathrm{m/s^2}$).

- Find the momentum when the particle hits the ground.
- What would be the earth' velocity upon impact?

$$\vec{a} = \frac{\vec{F}}{m} \leftrightarrow d\vec{v} = \vec{a}dt = \frac{\vec{F}dt}{m} \tag{2.110}$$

- or, equally true, they change the momentum of an object

$$\frac{d\vec{p}}{dt} = \vec{F} \leftrightarrow d\vec{p} = \vec{F}dt \tag{2.111}$$

Many physicists like the second bullet: forces change the momentum of an object, but for that they need time to act.

Momentum is a more fundamental concept in physics than acceleration. That is another reason why physicists prefer the second way of looking at forces.

Connecting physics and calculus

Let's look at a particle of mass m, that has initially (say at $t = 0$) a velocity v_0. For $t > 0$ the particle is subject to a force that is of the form $F = -bv$. This is a kind of frictional force: the faster the particle goes, the larger the opposing force will be.

We would like to know how the position of the particle is as a function of time.

We can answer this question by applying Newton 2:

$$m\frac{dv}{dt} = F \Rightarrow m\frac{dv}{dt} + bv = 0 \tag{2.112}$$

Clearly, we have to solve a differential equation which states that if you take the derivative of v you should get something like $-v$ back. From calculus we know, that exponential function have the feature that when we differentiate them, we get them back. So, we will try $v(t) = Ae^{-\mu t}$ with A and μ to be determined constants.

We substitute our trial v:

$$m \cdot A \cdot -\mu e^{-\mu t} + b \cdot Ae^{-\mu t} = 0 \tag{2.113}$$

This should hold for all t. Luckily, we can scratch out the term $e^{-\mu t}$, leaving us with:

$$-mA\mu + Ab = 0 \tag{2.114}$$

We see, that also our unknown constant A drops out. And, thus, we find

$$\mu = \frac{b}{m} \tag{2.115}$$

Next we need to find A: for that we need an initial condition, which we have: at $t = 0$ is $v = v_0$. So, we know:

$$v(t) = Ae^{-\frac{b}{m}t} and v(0) = v_0 \tag{2.116}$$

From the above we see: $A = v_0$ and our final solution is:

$$v(t) = v_0 e^{-\frac{b}{m}t} \tag{2.117}$$

Solution 2.61: Solution to Exercise 13

Let's do this one together. We follow the standard approach of IDEA: Interpret (and make your sketch!), develop (think 'model'), evaluate (solve your model) and assess (does it make any sense?).

Interpret

First a sketch: draw what is needed, no more, no less.

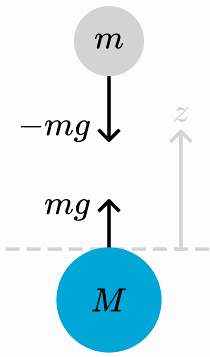


Figure 2.62: align: center

Develop

Actually this is half of the work, as when deciding what is needed we need to think what the problem really is. Above, is a sketch that could work. Both the object m and the earth (mass M) are drawn schematically. On each of them acts a force, where we know that on m standard gravity works. As a consequence of N3, a force equal in strength but opposite in direction acts on M.
Why do we draw forces? Well, the question mentions 'momentum the particle hits the ground'. Momentum and forces are coupled via N2.

We have drawn a z-coordinate: might be handy to remind us that this looks like a 1D problem (remember: momentum and force are both vectors).

As a first step, we ignore the motion of the earth. Argument? The magnitude of the ratio of the acceleration of earth over object is given by:

$$\frac{a_e}{a_o} = \frac{\mid F_{o \to e} \mid / m_e}{\mid F_{e \to o} \mid / m_o} = \frac{m_o}{m_e} \tag{2.106}$$

here for the second equality we used N3.

For all practical purposes, the mass of the object is many orders of magnitude smaller than that of the earth. Hence, we can conclude that the acceleration of the earth is many orders of magnitude less than that of the object. The latter is of the order of g, gravity's acceleration constant at the earth. Thus, the acceleration of the earth is next to zero and we can safely assume our lab system, that is connected to the earth, can be treated as an inertial system with, for us, zero velocity.

Evaluate

The remainder is straightforward. Now we have an object, that moves under a constant force. So its velocity will increase linearly in time:

$$\frac{dp}{dt} = -mg \Rightarrow p(t) = m\underbrace{v_0}_{=0} - mgt = -mgt \tag{2.107}$$

From the momentum, we can calculate the velocity and from the velocity the position:

$$v = -gt \Rightarrow \frac{dz}{dt} = -gt \Rightarrow z(t) = z_0 - \frac{1}{2}gt^2 = H - \frac{1}{2}gt^2 \tag{2.108}$$

Assess

We found that the particle changed its momentum from $p_i = 0$ to $p_f = -mv$. The earth compensates for this, to keep momentum conserved. That gave that earth got a tiny, tiny upwards velocity. We could estimate the displacement of the earth. Suppose, the particle has mass m=1kg and is dropped from a height $H = 100$m.

From the solution for v, we easily find the position of m as a function of time. Let's assume that the particle was in the origin at $t = 0$, thus $x(0) = 0$. So, we find for the position

$$\frac{dx}{dt} \equiv v = v_0 e^{-\frac{b}{m}t} \Rightarrow x = v_0 \cdot \left(-\frac{m}{b} e^{-\frac{b}{m}t}\right) + B \tag{2.118}$$

We find B with the initial condition and get as final solution:

$$x(t) = \frac{m v_0}{b}\left(1 - e^{-\frac{b}{m}t}\right) \tag{2.119}$$

If we inspect and assess our solution, we see: the particle slows down (as is to be expected with a frictional force acting on it) and eventually comes to a stand still. At that moment, the force has also decreased to zero, so the particle will stay put.

2.2.4.1 Inertia

Inertia is denoted by the letter m for mass. And mass is that property of an object that characterizes its resistance to changing its velocity. Actually, we should have written something like m_i, with subscript i denoting inertia.

Why? There is another property of objects, also called mass, that is part of Newton's Gravitational Law.

Two bodies of mass m_1 and m_2 that are separated by a distance r_{12} attract each other via the so-called gravitational force ($\hat{r}_{12}$ is a unit vector along the line connecting m_1 and m_2):

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r_{12}^2}\hat{r}_{12} \tag{2.120}$$

Here, we should have used a different symbol, rather than m. Something like m_g, as it is by no means obvious that the two 'masses' m_i and m_g refer to the same property. If you find that confusing, think about inertia and electric forces. Two particles with each an electric charge, q_1 and q_2, respectively exert a force on each other known as the Coulomb force:

$$\vec{F}_{C,12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12} \tag{2.121}$$

We denote the property associated with electric forces by q and call it charge. We have no problem writing

$$\begin{gathered}\vec{F} = m\vec{a} \\ \vec{F}_C = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}\end{gathered} \tag{2.122}$$

We do not confuse q by m or vice versa. They are really different quantities: q tells us that the particle has a property we call 'charge' and that it will respond to other charges, either being attracted to, or repelled from. How fast it will respond to this force of another charged particle depends on m. If m is big, the particle will only get a small acceleration; the strength of the force does not depend on m at all. So far, so good. But what about m_g? That property of a particle that makes it being attracted to another particle with this same property, that we could have called 'gravitational charge'. It is clearly different from 'electrical charge'. But would it have been logical that it was also different from the property inertial mass, m_i?

$$\begin{gathered}\vec{F} = m_i\vec{a} \\ \vec{F}_g = -G\frac{m_g M_g}{r^2}\hat{r}\end{gathered} \tag{2.123}$$

As far as we can tell (via experiments) m_i and m_g are the same. Actually, it was Einstein who postulated that the two are referring to the same property of an object: there is no difference.

Force field
We have seen, forces like gravity and electrostatics act between objects. When you push a car, the force is applied locally, through direct contact. In contrast, gravitational and electrostatic forces act over a distance — they are present throughout space, though they still depend on the positions of the objects involved.

One powerful way to describe how a force acts at different locations in space is through the concept of a **force field**. A force field assigns a force vector (indicating both direction and magnitude) to every point in space, telling you what force an object would experience if placed there.

For example, the graph below at the left shows a gravitational field, described by $\vec{F}_g = G\frac{mM}{r^2}\hat{r}$. Any object entering this field is attracted toward the central mass with a force that depends on its distance from that mass's center.

The figure on the right shows the force field that a positively charged particle would feel due to the presence of 2 negatively charged particles (both of the same charge). Clearly this is a much more complicated force field.

Measuring mass or force
So far we did not address how to measure force. Neither did we discuss how to measure mass. This is less trivial than it looks at first side. Obviously, force and mass are coupled via N2: $F = ma$.

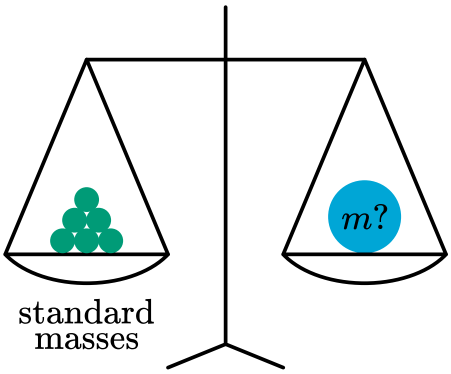


Figure 2.63: Can force be measured using a balance?

The acceleration can be measured when we have a ruler and a clock, i.e. once we have established how to measure distance and how to measure time intervals, we can measure position as a function of time and from that velocity and acceleration.

But how to find mass? We could agree upon a unit mass, an object that represents by definition 1kg. In fact we did. But that is only step one. The next question is: how do we compare an unknown mass to our standard. A first reaction might be: put them on a balance and see how many standard kilograms you need (including fractions of it) to balance the unknown mass. Sounds like a good idea, but is it? Unfortunately, the answer is not a 'yes'.

As on second thought: the balance compares the pull of gravity. Hence, it 'measures' gravitational mass, rather than inertia. Luckily, Newton's laws help. Suppose we let two objects, our standard mass and the unknown one, interact under their mutual interaction force. Every other force is excluded. Then, on account on N2 we have

$$\begin{cases} m_1 a_1 = F_{21} \\ m_2 a_2 = F_{12} = -F_{21} \end{cases} \tag{2.124}$$

where we used N3 for the last equality. Clearly, if we take the ratio of these two equations we get:

$$\frac{m_1}{m_2} = \mid \frac{a_2}{a_1} \mid \tag{2.125}$$

irrespective of the strength or nature of the forces involved. We can measure acceleration and thus with this rule express the unknown mass in terms of our standard.

Note

We will not use this method to measure mass. We came to the conclusion that we can't find any difference in the gravitational mass and the inertial mass. Hence, we can use scales and balances for all practical purposes. But the above shows, that we can safely work with inertial mass: we have the means to measure it and compare it to our standard kilogram.

Now that we know how to determine mass, we also have solved the problem of measuring force. We just measure the mass and the acceleration of an object and from N2 we can find the force. This allows us to develop 'force measuring equipment' that we can calibrate using the method discussed above.

Intermezzo: kilogram, unit of mass

In 1795 it was decided that 1 gram is the mass of 1 cm of water at its melting point. Later on, the kilogram became the unit for mass. In 1799, the kilogramme des Archives was made, being from then on the prototype of the unit of mass. It has a mass equal to that of 1 liter of water at 4°C (when water has its maximum density).The International Prototype of the Kilogram, whose mass was defined to be one kilogram from 1889 to 2019. Picture by BIPM, CC BY-SA 3.0 igo, https://commons.wikimedia.org/w/index.php?curid=117707466In recent years, it became clear that using such a standard kilogram does not allow for high precision: the mass of the standard kilogram was, measured over a long time, changing. Not by much (on the order of 50 micrograms), but sufficient to hamper high precision measurements and setting of other standards. In modern physics, the kilogram is now defined in terms of Planck's constant. As Planck's constant has been set (in 2019) at exactly , the kilogram is now defined via , the meter and second.

2.2.4.2 Eötvös experiment on mass

The question whether inertial mass and gravitational mass are the same has put experimentalists to work. It is by no means an easy question. Gravity is a very weak force. Moreover, determining that two properties are identical via an experiment is virtually impossible due to experimental uncertainty. Experimentalist can only tell the outcome is 'identical' within a margin. Newton already tried to establish experimentally that the two forms of mass are the same. However, in his days the inaccuracy of experiments was rather large. Dutch scientist Simon Stevin concluded in 1585 that the difference must be less than 5%. He used his famous 'drop masses from the church' experiments for this (they were primarily done to show that every mass falls with the same acceleration).

A couple of years later, Galilei used both fall experiments and pendula to improve this to: less than 2%. In 1686, Newton using pendula managed to bring it down to less than 1‰ .

An important step forward was set by the Hungarian physicist, Loránd Eötvös (1848-1918). We will here briefly introduce the experiment. For a full analysis, we need knowledge about angular momentum and centrifugal forces that we do not deal with in this book.

The experiment

The essence of the Eötvös experiment is finding a set up in which both gravity (sensitive to the gravitational mass) and some inertial force (sensitive to the inertial mass) are present. Obviously, gravitational forces between two objects out of our daily life are extremely small. These will be very difficult to detect and thus introduce a large error if the experiment relies on measuring them. Eötvös came up with a different idea. He connected two different objects with different masses, m_1 and m_2, via a (almost) massless rod. Then, he attached a thin wire to the rod and let it hang down.

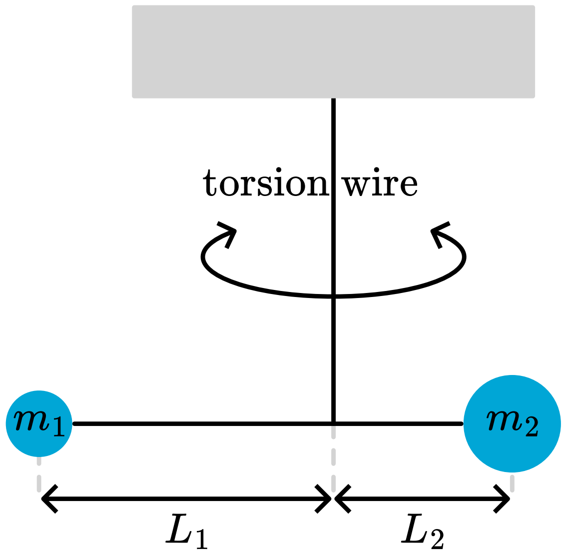


Figure 2.64: align: center :alt: Mass m1 and m2 are connected to either end of a horizontal rod. The rod, in turn, is connected by a vertical wire to the ceiling. The rod can rotate around its suspension point.

Torsion balance used by Eötvös.

This is a sensitive device: any mismatch in forces or torques will have the setup either tilt or rotate a bit. Eötvös attached a tiny mirror to one of the arms of the rod. If you shine a light beam on the mirror and let it reflect and be projected on a wall, then the smallest deviation in position will be amplified to create a large motion of the light spot on the wall.

In Eötvös experiment two forces are acting on each of the masses: gravity, proportional to m_g, but also the centrifugal force $F_c = m_i R\omega^2$, the centrifugal force stemming from the fact that the experiment is done in a frame of reference rotating with the earth. This force is proportional to the inertial mass. The experiment is designed such that if the rod does not show any rotation around the vertical axis, then the gravitational mass and inertial mass must be equal. It can be done with great precision and Eötvös observed no measurable rotation of the rod. From this he could conclude that the ratio of the gravitational over inertial mass differed less from 1 than $5 \cdot 10^{-8}$. Currently, experimentalist have brought this down to $1 \cdot 10^{-15}$.

Note

The question is not if m_i/m_g is different from 1. If that was the case but the ratio would always be the same, then we would just rescale m_g, that is redefine the value of the gravitational const G to make m_g equal to m_i. No, the question is whether these two properties are separate things, like mass and charge. We can have two objects with the same inertial mass but give them very different charges. In analogy: if m_i and m_g are fundamentally different quantities then we could do the same but now with inertial and gravitational mass.

Tip

Want to know more about this experiment? Watch this videoclip.

2.2.5 Examples, exercises and experiments

Updated: 04 feb 2026 Here are some examples and exercises that deal with forces. Make sure you practice IDEA.

2.2.5.1 Worked Examples

2.2.5.1.1 Slowing a mass down

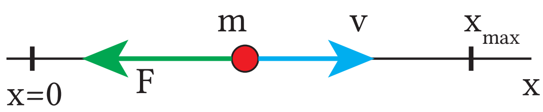
In this example we will consider a 1-dimensional case: a mass, m has initial velocity v_0. From $t = 0$ onwards a force F is acting on m, slowing it down. Eventually, m will come to a stand still.

The final question to be answered is: what is the distance m has traveled from $t = 0$ until m has stopped moving?

We will inspect the simplest case: the force is constant and has a magnitude $\mu m g$ (with μ a positive constant). This is one of the simplest frictional forces. It is proportional to the weight of m and is a first order approximation for a mass sliding over a horizontal plane.

Interpret the problem

First we make a sketch and draw what is relevant for this problem.



The problem is 1-dimensional, so we only need one coordinate. Hence we have drawn the x-axis. The mass is somewhere on the axis and has at that position velocity v. We also draw that, as the velocity will change. Moreover, velocity is related to momentum ($p = mv$) and -as a force is acting on m- we expect that we will use N2.

We also draw the force. As the force is slowing down the mass, it will have to act in the direction opposite to the velocity. That means in our case: F points in the negative x-direction. Finally, we indicate the position where the mass will stop moving: x_{max}.

We conclude our interpret-phase with our idea on how to approach this problem:

- a force is slowing down m: that calls for setting up N2
- at some point in time m has zero velocity: we need to find that time (let's call it t_f). We can do that via N2.
- we need to find the trajectory of m, i.e. $x(t)$ and than substitute t_f to find $x_{max} = x(t_f)$.

Develop the solution

Let's set up N2:

$$m\frac{dv}{dt} = F \tag{2.126}$$

This is a first-order differential equation and we need 1 initial condition. Given is:

$$t = 0 \rightarrow v = v_0 \tag{2.127}$$

Now, we have our model for $v(t)$. Once we solved it, we can use this solution to find the trajectory, $x(t)$. For that step we use $v \equiv \frac{dx}{dt}$, but in reversed order, as we like to treat it now as a differential equation for $x(t)$:

$$\frac{dx}{dt} = v(t) \tag{2.128}$$

with initial condition

$$t = 0 \rightarrow x = 0 \tag{2.129}$$

Actually, this is our own choice: we have taken the origin of the x-axis as the position where m is at $t = 0$. We could have taken any other point, but this is just convenient.

Evaluate the solution

We start with solving N2:

$$m\frac{dv}{dt} = F \rightarrow \frac{dv}{dt} = -\frac{\mu mg}{m} = -\mu g \tag{2.130}$$

Note: we have now explicitly given the minus-sign as the force is acting in the negative x-direction. Moreover, the friction constant $\mu > 0$. This differential equation is easy to solve, as the right hand side is a constant:

$$v(t) = -\mu gt + C_1 \tag{2.131}$$

with C_1 an integration constant. We can find this constant by looking at the initial condition:

$$t = 0 \rightarrow v = v_0 \Rightarrow v_0 = -\mu g0 + C_1 \Rightarrow C_1 = v_0 \tag{2.132}$$

Thus, the solution for $v(t)$ is:

$$v(t) = v_0 - \mu gt \tag{2.133}$$

From this equation, we can find when the mass stops moving:

$$v(t_f) = 0 \Rightarrow 0 = v_0 - \mu gt_f \Rightarrow t_f = \frac{v_0}{\mu g} \tag{2.134}$$

Now we are ready to find the trajectory $x(t)$:

$$\frac{dx}{dt} = v(t) = v_0 - \mu gt \Rightarrow x(t) = v_0 t - \frac{1}{2}\mu gt^2 + C_2 \tag{2.135}$$

and we use the initial condition to find C_2:

$$t = 0 \rightarrow x = 0 \Rightarrow 0 = v_{00} - \frac{1}{2}\mu g0^2 + C_2 \Rightarrow C_2 = 0 \tag{2.136}$$

Thus, the solution for the trajectory is:

$$x(t) = v_0 t - \frac{1}{2}\mu gt^2 \tag{2.137}$$

Finally, we can find x_{max} by substituting t_f into $x(t)$:

$$\begin{aligned} x_{max} = x(t_f) &= v_0\frac{v_0}{\mu g} - \frac{1}{2}\mu g\left(\frac{v_0}{\mu g}\right)^2 \\ &= \frac{v_0^2}{\mu g} - \frac{1}{2}\frac{v_0^2}{\mu g} = \frac{1}{2}\frac{v_0^2}{\mu g} \end{aligned} \tag{2.138}$$

Assess the solution

The final result makes sense: the distance traveled increases with increasing initial velocity and decreases with increasing friction coefficient μ and gravitational acceleration g. Furthermore, $x_{max} > 0$ as it should: the mass is moving from $x = 0$ into the positive x-direction.

Exercise 2.66: Force on a particle 🌶️

Consider a point particle of mass m, moving at a velocity v_0 along the x-axis. At $t = 0$ a constant force acts on the particle in the negative x-direction. The force lasts for a small time interval Δt.

What is the strength of the force, if it brings the particle exactly to a zero-velocity? Start by making a drawing.

Also, the units match: x_{max} has SI-units [m] and thus, the right hand side of the equation must also be in [m]. We can check this easily:

$$
\begin{gathered}
[v_0] \rightarrow \frac{m}{s} \\
\rightarrow - \\
\rightarrow \frac{m}{s^2}\} \Rightarrow \left[\frac{v_0^2}{\mu g}\right] = \frac{m^2/s^2}{m/s^2} = m
\end{gathered}
\tag{2.139}
$$

Indeed, our answer is dimensionally correct.

Note: the mass is irrelevant in this example. That is a consequence of the force being proportional to m. Hence, as we saw, m drops out of N2. In general that is of course not the case.

Finally: in this example $v(t)$ is a linear function in t whereas $x(t)$ is a parabola. This is rather the exception than the rule! It only happens when the force F is a constant, that is it doesn't change during the process. We will come across plenty of cases where F is not a constant. It may vary explicitly with time, but in most cases it will change with the position of the mass. So keep in mind: $x(t) = x_0 + v_0 t + \frac{1}{2}at^2$ is *only* the solution if the force acting is a true constant.

2.2.5.2 Exercises set 1

2.2.5.3 Answers set 1

2.2.5.4 Exercises set 2

```
interactive(children=(FloatSlider(value=0.7853981633974483,
description='theta', max=1.5707963267948966, min=0…
```

```
<function __main__.update(theta, F_girl)>
```

```
interactive(children=(FloatSlider(value=0.7853981633974483,
description='theta', max=1.5707963267948966, step=…
```

```
<function __main__.update(theta, mu)>
```

```
interactive(children=(IntSlider(value=1, description='force_num', max=3,
min=1), Output()), _dom_classes=('wid…
```

```
<function __main__.update(force_num)>
```

```
interactive(children=(FloatSlider(value=9.81, description='g (m/s²)',
max=15.0, min=1.5), IntSlider(value=1, d…
```

```
<function __main__.run_animation(g=9.81, M=1)>
```

2.2.5.5 Experiments

Warning

Under construction

Friction

Exercise 2.67: Shooting a ball 🌶️

A ball is shot from a 10m high hill with a velocity of 10m/s under an angle of 30^o, see Figure 2.

1. How long is the ball in the air?
2. How far does it travel in the horizontal direction?
3. With what velocity does the ball hit the ground?

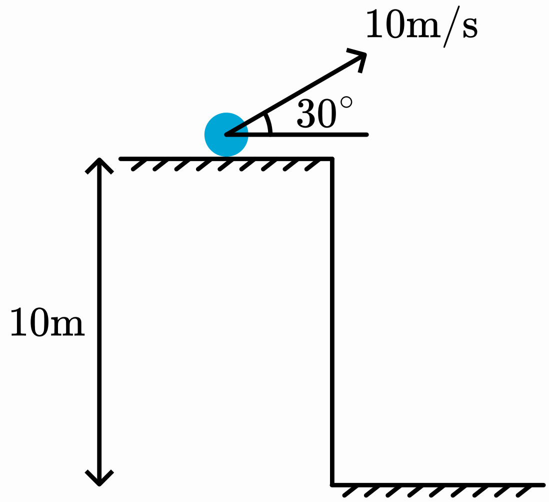


Figure 2.68: A ball on a hill launched under an angle.

Pick various (note)books with different covers (smooth, rough) and do the following:Push the book against the wall, first horizontally then slowly change the direction upwards. Note the force needed to keep the book in place.What is the difference between the rough and smooth cover? When does your finger start to slide?

some experiment

IOlab push it upwards, see v,t. a,t

Exercise 2.69: Constant force on a particle 🌶️

A particle of mass m moves along the x-axis. At time $t = 0$ it is at the origin with velocity v_0. For $t > 0$, a constant force acts on the particle. This is a 1-dimensional problem.

- Derive the acceleration of the particle as a function of time.
- Derive the velocity of the particle as a function of time.
- Derive the position of the particle as a function of time.

Exercise 2.70: Time dependent force on a particle 🌶️

A particle of mass m moves along the x-axis. At time $t = 0$ it is at the origin with velocity v_0. For $t > 0$ the particle is subject to a force $F_0 \sin(2\pi f_0 t)$. This is a 1-dimensional problem.

- Calculate the acceleration of the particle as a function of time.
- Calculate the velocity of the particle as a function of time.
- Calculate the position of the particle as a function of time.

Exercise 2.71: Particle trajectory 🌶️

A particle follows a straight path with a constant velocity. At $t = 0$ the particle is at point A with coordinate $(0, y_A)$, while at $t = t_1$ it is at B with coordinate $(x_B, 0)$. The coordinates are given in a Cartesian system. The problem is 2-dimensional.

1. Make a sketch.
2. Find the position of the particle at arbitrary time $0 < t < t_1$.
3. Derive the velocity of the particle from position as function of time.

Represent vectors in a Cartesian coordinate system using the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

Exercise 2.72: Different coordinate systems 🌶️ 🌶️

In Classical Mechanics we often use a coordinate system to describe motion of object. In this exercise, you will look at two Cartesian coordinate systems. System S has coordinates (x, y) and corresponding unit vectors $\hat{x}$ and $\hat{y}$.
The second system, S', uses (x', y') and corresponding unit vectors. The x'-axis makes an angle of 30° with the x-axis (measured counter-clockwise).

1. Make a sketch.
2. Determine the relations between $\hat{x}'$ and $\hat{x}, \hat{y}$ as well as between $\hat{y}'$ and $\hat{x}, \hat{y}$

An object has, according to S, a velocity of $\vec{v} = 3\hat{x} + 5\hat{y}$.

1. Determine the velocity according to S'.

Atwood machine

Exercise 2.73: Rotating unit vectors 🌶️

According to your observations, a particle is located at position (1,0) (you use a Cartesian coordinate system). The particle has no velocity and no forces are acting on it.
Another observer, S', uses a Cartesian coordinate system described by (x', y'). You notice that her unit vectors rotate at a constant speed compared to your unit vectors:

$$\hat{x}' = \cos(2\pi f t)\hat{x} + \sin(2\pi f t)\hat{y} \tag{2.140}$$

$$\hat{y}' = -\sin(2\pi f t)\hat{x} + \cos(2\pi f t)\hat{y} \tag{2.141}$$

1. Find the position of the particle according to the other observer, S'.
2. Calculate the velocity of the particle according to S'.

Exercise 2.74: Moving over a frictionless table🌶️

A particle of mass m moves at a constant velocity v_0 over a frictionless table. The direction it is moving in, is at 45° with the positive x-axis. At some point in time, the particle experiences a force $\vec{F} = -b\vec{v}$ with $b > 0$.
We call this time $t = 0$ and take the position of the particle at that time as our origin.

1. Make a sketch.
2. Determine whether this problem needs to be analyzed as a 1D or a 2D problem.
3. Set up N2 in the form $m\frac{d\vec{v}}{dt} = \vec{F}$
4. Solve N2 and find the velocity of the particle as a function of time.
5. What happens to the particle for large t?

Exercise 2.75: Parabolic trajectory with maximum area[4]

A ball is thrown at speed v from zero height on level ground. We want to find the angle θ at which it should be thrown so that the area under the trajectory is maximized.

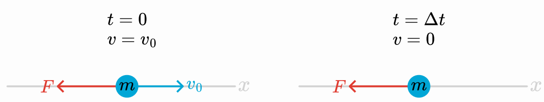
1. Sketch of the trajectory of the ball.
2. Use dimensional analysis to relate the area to the initial speed v and the gravitational acceleration g.
3. Write down the x and y coordinates of the ball as a function of time.
4. Find the total time the ball is in the air.
5. The area under the trajectory is given by $A = \int y \mathrm{d}x$. Make a variable transformation to express this integral as an integration over time.
6. Evaluate the integral. Your answer should be a function of the initial speed v and angle θ.
7. From your answer at (6), find the angle that maximizes the area, and the value of that maximum area.

Exercise 2.76: Two attracting particles[5]

Two particles on a line are mutually attracted by a force $F = -ar$, where a is a constant and r the distance of separation. At time $t = 0$, particle A of mass m is located at the origin, and particle B of mass $m/4$ is located at $r = 5.0$ cm. Both particles have zero velocity at $t = 0$. If the particles are at rest at $t = 0$, at what value of r do they collide?

A modern version of the Atwood machine uses an airtrack, as given in . A glider is connected via a string that runs over a low-friction pulley. The masses can be changed by adding small weights.Using a horizontal track, the acceleration is said to be uniform. Verify this and show that the acceleration is given by .Now change the mass such that and verify this.If we would change the angle of the airtrack so that we have a slope, the acceleration will decrease first and subsequently reverse direction. At what angle will the acceleration be once again? Verify this experimentally.

Solution 2.77: Solution to Exercise 1



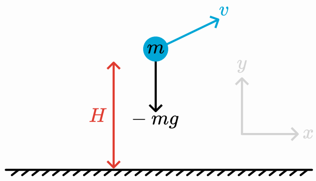
$$\vec{F} = -\frac{m v_0}{\Delta t}\hat{x}$$

[4]Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

[5]Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Solution 2.79: Solution to Exercise 2

Interpret



Develop

We know $v_y = v\sin(\theta)$ and $v_x = v\cos(\theta)$.

The motion of the ball can be spilt in two components: horizontal, i.e. x-direction, and vertical, tha is y-direction.

In the vertical direction gravity acts: $F_y = -mg$. Thus the equation of motion in the y-direction is: $ma_y = F_y = -mg.Theverticalpositioncanthusbeexpressedas$ s_y(t)=s_{y0}+v_{y0}t-1/2gt^2$.

In the horizontal direction no force is active, thus: $ma_x = 0 \rightarrow s_x(t) = s_{x0} + v_{x0}t$

The magnitude of the velocity of the ball hitting the ground can be expressed in terms of v_x and v_y as $v_e = \sqrt{v_x^2 + v_y^2}$

Evaluate

We have as initial velocity: $v_{y0} = v\sin(\theta) = 10 * \sin(30) = 5m/s$
$v_{x0} = v\cos(\theta) = 10 * \cos(30) = 5\sqrt{3}m/s$

Solving $s_y(t) = s_{y0} + v_{y0}t - 1/2gt^2$ for $s_y = 0$ with $s_{y0} = H$ gives for the time the ball is in the air:

$$t_{air} = \frac{v_{y0}}{g} + \sqrt{\frac{v_{y0}^2}{g^2} + \frac{2H}{g}} = 2.77s \tag{2.142}$$

Next, we realize that $v_x = const = v_{x0}$ as there is no force acting in the x-direction. Thus the horizontal distance traveled is

$$\Delta x = v_{X0} t_{air} = 24.0 \text{m}

For the velocity when hitting the ground is (that is, its magnitude), we need both the x and y-component:

$$\begin{aligned} v_x &= v_{x0} = 8.66m/s \\ v_y &= v_{0y} - gt \rightarrow v_y(t_{air}) = \sqrt{v_{y0}^2 + 2gH} = 14.9m/s \end{aligned} \tag{2.143}$$

$$v_{ground} = \sqrt{v_x^2 + v_y^2} = \sqrt{v_{x0}^2 + v_y^2} = 17.2m/s \tag{2.144}$$

Assess

The velocity upon impact is larger than the initial velocity. This makes sense. The ball first travels upwards, then downwards and will pass $s_y = H$ again on the downward motion. Then it will further accelerate to the ground and thus have a larger y-component of the velocity than at the start.

Solution 2.81: Solution to Exercise 3

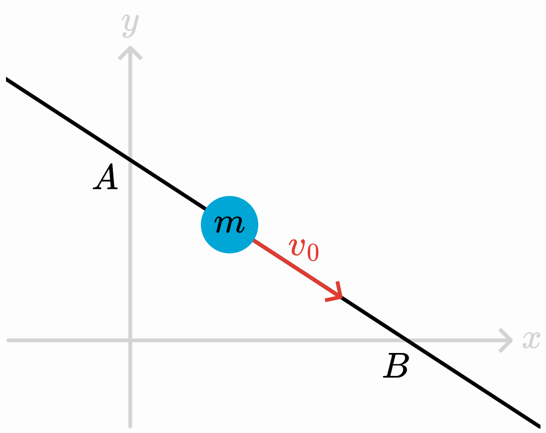
1. $a = \frac{F}{m}$ is constant
2. $v(t) = v_0 + at$
3. $x(t) = v_0 t + \frac{1}{2}at^2$

Solution 2.82: Solution to Exercise 4

1. $a = \frac{F}{m} = \frac{F_0}{m}\sin(2\pi f_0 t)$ is **not** constant
2. $v(t) = v_0 + \frac{F_0}{2\pi f_0 m}(1 - \cos(2\pi f_0 t))$
3. $x(t) = v_0 t + \frac{F_0}{2\pi f_0 m}t - \frac{F_0}{4\pi^2 f_0^2 m}\sin 2\pi f_0 t$

Solution 2.83: Solution to Exercise 5

1.



2. Particle moves at constant velocity, thus path is a straight line:

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t = x_0\hat{\imath} + y_0\hat{\jmath} + v_{0x}t\hat{\imath} + v_{0y}t\hat{\jmath} \tag{2.145}$$

At $t = 0 : \vec{r}(0) = 0\hat{\imath} + y_A\hat{\jmath} \to \vec{r}(0) = \vec{r}_0 = 0\hat{\imath} + y_A\hat{\jmath} \to x_0 = 0 and y_0 = y_A$

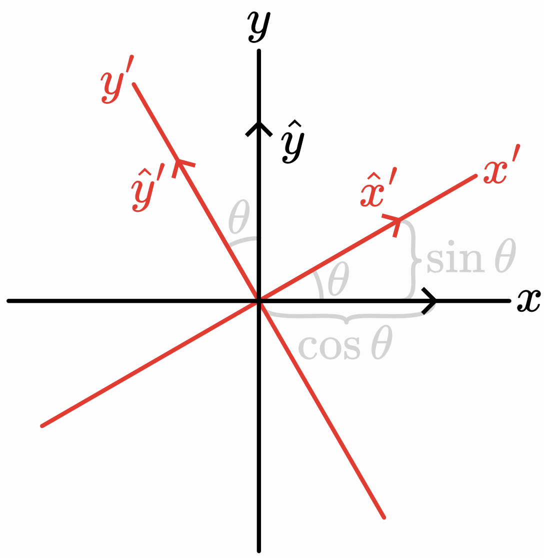
At $t = t_1$:

$$\begin{aligned} \vec{r}(t_1) &= x_B\hat{\imath} + 0\widehat{()}\jmath \to \\ \vec{r}(t_1) &= \vec{r}_0 + \vec{v}_0 t_1 \\ &= (0 + v_{0x}t_1)\hat{\imath} + (y_A + v_{0y}t_1)\hat{\jmath} \to \\ v_{0x} &= \frac{x_B}{t_1} and v_{0y} = -\frac{y_A}{t_1} \end{aligned} \tag{2.146}$$

3. Thus, we find $\vec{v} = \frac{x_B}{t_1}\hat{\imath} - \frac{y_A}{t_1}\hat{\jmath}$

Solution 2.85: Solution to Exercise 6

1.



2.

$$\begin{aligned}\hat{x}' &= \cos\theta\hat{x} + \sin\theta\hat{y} = \frac{1}{2}\sqrt{3}\hat{x} + \frac{1}{2}\hat{y} \\ \hat{y}' &= -\sin\theta\hat{x} + \cos\theta\hat{y} = -\frac{1}{2}\hat{x} + \frac{1}{2}\sqrt{3}\hat{y}\end{aligned} \tag{2.147}$$

2. Invert:

$$\begin{aligned}\hat{x} &= \cos\theta\hat{x}' - \sin\theta\hat{y}' = \frac{1}{2}\sqrt{3}\hat{x}' - \frac{1}{2}\hat{y}' \\ \hat{y} &= \sin\theta\hat{x}' + \cos\theta\hat{y}' = \frac{1}{2}\hat{x}' + \frac{1}{2}\sqrt{3}\hat{y}'\end{aligned} \tag{2.148}$$

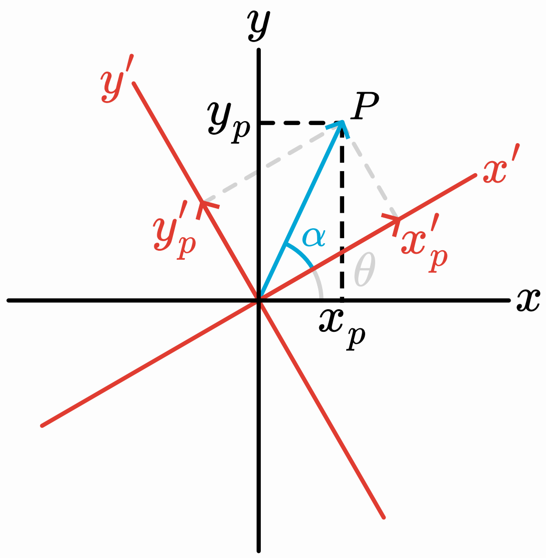
velocity:

$$\begin{aligned}\vec{v} &= v_x\hat{x} + v_y\hat{y} \\ &= v_x(\cos\theta\hat{x}' - \sin\theta\hat{y}') + v_y(\sin\theta\hat{x}' + \cos\theta\hat{y}') \\ &= \left(v_x\cos\theta + v_y\sin\theta\right)\hat{x}' + \left(-v_x\sin\theta + v_y\cos\theta\right)\hat{y}'\end{aligned} \tag{2.149}$$

from which we find

$$\vec{v} = \left(\frac{3}{2}\sqrt{3} + \frac{5}{2}\right)\hat{x}' + \left(-\frac{3}{2} + \frac{5}{2}\sqrt{3}\right)\hat{y}' \tag{2.150}$$

Solution 2.87: Solution to Exercise 7



$$\begin{aligned}\hat{x}' &= \cos(2\pi f t)\hat{x} + \sin(2\pi f t)\hat{y} \\ \hat{y}' &= -\sin(2\pi f t)\hat{x} + \cos(2\pi f t)\hat{y}\end{aligned} \tag{2.151}$$

The unit vectors of S' rotate with a frequency f with respect to the unit vectors of S. This means, that the coordinate system of S' rotates: the rotation angle is a function of time, i.e. $\theta(t) = 2\pi f t$

From the figure we see, that the coordinates of a point P, (x_p, y_p) according to S, are related to those used by S', $({x'}_p, {y'}_p)$ via:

$$\begin{aligned}x_p &= OP\cos(\alpha+\theta) = OP(\cos\alpha\cos\theta - \sin\alpha\sin\theta) = {x'}_p\cos\theta - {y'}_p\sin\theta \\ y_p &= OP\sin(\alpha+\theta) = OP(\cos\alpha\sin\theta + \sin\alpha\cos\theta) = {x'}_p\sin\theta + {y'}_p\cos\theta\end{aligned} \tag{2.152}$$

or written as the coordinate transformation:

$$\begin{aligned}x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta\end{aligned} \tag{2.153}$$

with its inverse

$$\begin{aligned}x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta\end{aligned} \tag{2.154}$$

Note that in this case $\theta = 2\pi f t$, that is: it is a function of t.

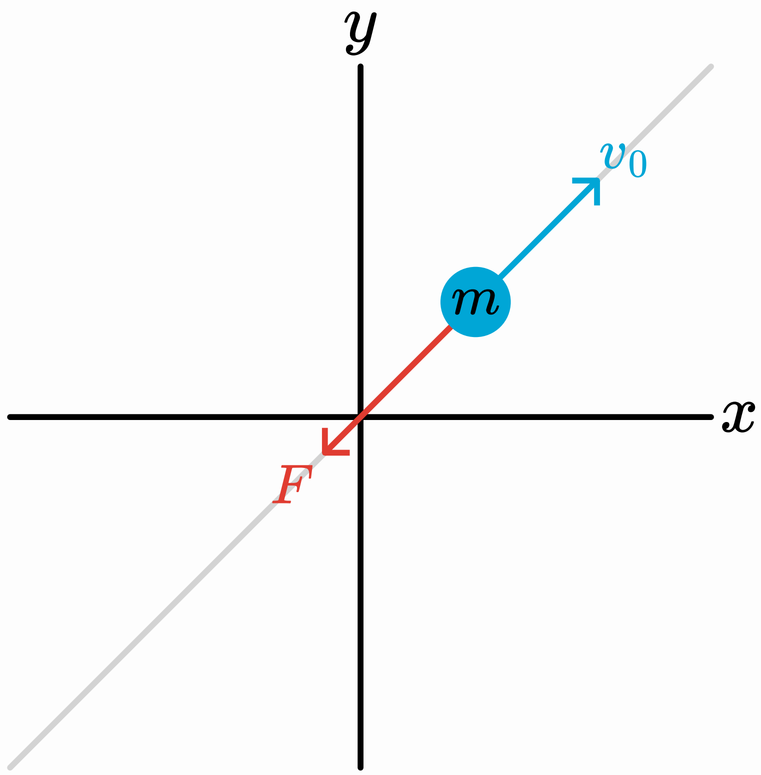
a) From the above relation we find that the point (1,0) in S will be denoted by S' as $(x'(t), y'(t)) = (\cos(2\pi f t), -\sin(2\pi f t))$

b) the velocity of the point (1,0) in S is according to S of course zero: $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$ S' will say:

$$\begin{aligned}x'(t) &= \cos(2\pi f t) \to \frac{dx'}{dt} = -2\pi f\sin(2\pi f t) \\ y'(t) &= -\sin(2\pi f t) \to \frac{dy'}{dt} = 2\pi f\cos(2\pi f t)\end{aligned} \tag{2.155}$$

Solution 2.89: Solution to Exercise 8

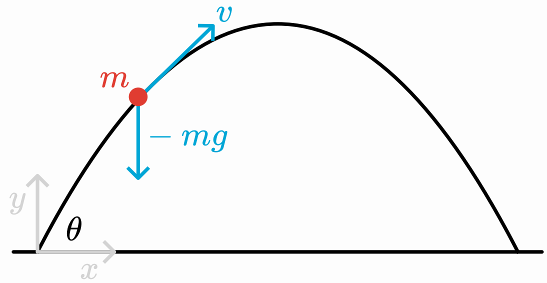
1.



2. Since $\vec{v}_0$ and $\vec{F}$ are parallel, the particle will not deviate from the line x=y. Hence, we are dealing with a 1-dimensional problem. The original coordinate system, (x, y), is not wrong: it is just not handy as it makes the problem look like 2D. Thus, we change our coordinate system, such that the new x-axis coincides with the original x=y line.
3. N2: $m\frac{dv}{dt} = -bv$ with initial conditions: $t = 0 \to x = 0$ and $t = 0 \to v = v_0$
4. $\frac{dv}{dt} - \frac{b}{m}v = 0 \to v = Ae^{-\frac{b}{m}t}$ initial condition: $t = 0 \to v = v_0 \Rightarrow A = v_0$ Thus: $v(t) = v_0 e^{-\frac{b}{m}t}$
5. for $t \to \infty : v \to 0$. The particle comes to rest and then, obviously, the friction force is zero.

Solution 2.91: Solution to Exercise 9

1.



2. We expect that the area, A, under the trajectory of the ball is a function of v, g, and θ. In a dimensional analysis we write this as 'product of powers':

$$A = v^a \cdot g^b \cdot \theta^c \tag{2.156}$$

and we make this expression dimensional correct. (Note: we don't mean that the final outcome of a full analysis is a product of powers, it can be any function but the units should be related in the right way and that is what this 'trick' with powers ensures.)

The area has units m^2, velocity m/s, g m/s^2 and θ is dimensionless (radians don't count as a dimension or unit). Thus:

$$\begin{aligned} m &: 2 = a + b \\ s &: 0 = -a - 2b \end{aligned} \tag{2.157}$$

This yields: $a = 4, b = -2$. Thus on dimensional grounds we may expect: $A \sim \frac{v^4}{g^2}$.

3. In the x-direction: no forces, hence $m\frac{v_x}{dt} = 0 \rightarrow x(t) = v\cos\theta t$

In the y-direction: $m\frac{dv_y}{dt} = -mg \rightarrow y(t) = v\sin\theta t - \frac{1}{2}gt^2$. Where we have used the initial conditions: $x(o) = 0, y(0) = 0, v_x(0) = v\cos\theta, v_y(0) = v\sin\theta$

4. Total time in the air: $v_y(t^*) = 0 \rightarrow t^* = \frac{2v}{g}\sin\theta$

5+6. Evaluate the area under the trajectory:

$$\begin{aligned} A &= \int_0^{x_{max}} ydx \\ &= \int_0^{t^*} \left(v\sin\theta t - \frac{1}{2}gt^2\right) v\cos\theta dt \\ &= v^2\sin\theta\cos\theta\frac{1}{2}(t^*)^2 - /frac16gv\cos\theta(t^*)^3 \\ &= \frac{2}{3}\frac{v^4}{g^2}\cos\theta\sin^3\theta \end{aligned} \tag{2.158}$$

7. We maximize the function $f(\theta) = \cos\theta\sin^3\theta$:

$$\frac{df}{d\theta} = \sin^2\theta(-\sin^2\theta + 3\cos^2\theta) \tag{2.159}$$

$$\frac{df}{d\theta} = 0 \rightarrow \sin\theta = 0 or \sin^2\theta = 3\cos^2\theta \tag{2.160}$$

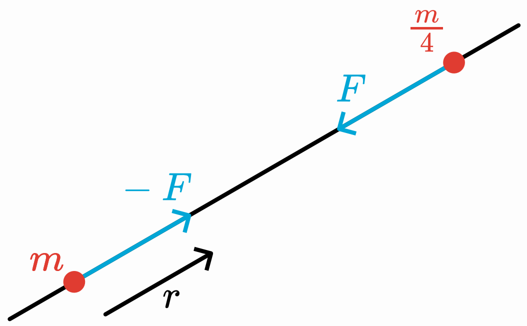
The first solution give a minimum for the area ($A = 0$). So we need the second solution:

$$\frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta = 3 \rightarrow \tan\theta = \sqrt{3} \rightarrow \theta = \frac{\pi}{3} \tag{2.161}$$

Solution 2.93: Solution to Exercise 10

Interpret

We start with a sketch.



This is a 1-dimensional problem. We will use r as the coordinate. Moreover, it is a problem involving two particles, that both can move. This makes it more difficult than 1-dimensional cases with only one particle.

Develop

We have to set up two equations of motion, one for particle 1 with mass m and position r_1 and one for particle 2 with mass $m/4$ and position r_2. When doing so, we should realize that the mutual force obeys Newton's third law: $F_{12} = -F_{21}$

$$\begin{aligned} m\frac{dv_1}{dt} &= a(r_2 - r_1) \\ \frac{m}{4}\frac{dv_2}{dt} &= -a(r_2 - r_1) \end{aligned} \tag{2.162}$$

We see that the two equations are coupled: we can't solve one without information from the other.

Evaluate

So, how do we proceed? First, let's think about the question. We are not asked to solve the equation of motion and find the trajectory. What we need to find is the position of the collision.

From the two equation of motion we can find important information about the velocities of both particles. Just add to two equations:

$$m\frac{dv_1}{dt} + \frac{m}{4}\frac{dv_2}{dt} = 0 \to \frac{dv_1}{dt} = -\frac{1}{4}\frac{dv_2}{dt} \tag{2.163}$$

Since both particles start rest, we find from the last equation: $v_1 = -\frac{1}{4}v_2$ at any time. Thus particle 2 will travel 4 times a distance than particle 1 in the same time interval. Consequently: if particle 1 has moved 1cm, particle 2 has moved 4cm. Thus the particles (originally separated by 5cm) will collide at $r = 1$cm.

Assess

It makes sense that the heavy particle has traveled less than the light one: they both feel at any moment the same force (apart from a sign). The light particle will accelerate faster than the heavy one. Moreover, they should collide somewhere on the line element originally separating them as they are attracted to each other.

We found both these elements in our solution.

Exercise 2.95: Who is strongest? 🌶

Who is strongest? Two strong boys try to keep a rope straight by each pulling hard at one end. A not so strong third person is pulling in the middle of the rope, but at an angle of 90° to the rope. The two strong boys have the task to keep the deviation of the rope to a small value, set by you.

How does the force and the angle depends on the force exerted by the girl?

Figure 2.96: Picture taken from Show the Physics

Exercise 2.97: Dropping a stone from a church tower 🌶

You drop a stone from a height of 50m the tower of the church. Calculate the velocity of the stone when it hits the ground (ignore friction). In the video you will see on the left a quick and dirty solution, NOT using IDEA. The right hand side uses IDEA and Newton's 2^{nd} law.

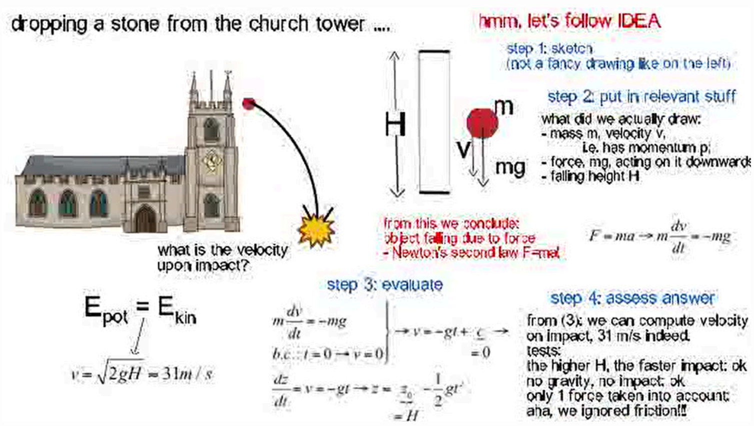


Figure 2.98: The worked out exercise

Exercise 2.99: Sliding down a slope 🌶

Two point particles slide down a slope: one feels friction the other doesn't. Can you analyse the situation and understand the graphs?

Exercise 2.100: 🌶️

Below are three forces and their resultant (v, t)- and (s, t)-diagrams. What kind of forces are acting?

Exercise 2.101: 🌶️

A mass $m = 1$kg (the red one in the drawing) is attached to a massless string. The string can move freely over a massless pulley. At the other end of the string a variable mass M (the grey one) is hanging. At $t = 0$ mass m is released, while the string is stretched to its full length.

The graph on the right side of the screen shows the velocity of m as a function of time.

- 'Play' with the acceleration and mass M, predict every time first what will happen to the motion.
- Describe the motion of m and M.
- Write down Newton's equation of motion for m and for M.

Exercise 2.102: 🌶️ 🌶️ 🌶️

A point particle (mass m) is from position $z = 0$ shot with a velocity v_0 straight upwards into the air. On this particle only gravity acts, i.e. friction with the air can be ignored. The acceleration of gravity, g, may be taken as a constant.

The following questions should be answered.

- What is the maximum height that the particle reaches?
- How long doe it take to reach that highest point?

Solve this exercise using IDEA.

- Sketch the situation and draw the relevant quantities.
- Reason that this exercise can be solved using $\vec{F} = m\vec{a}$ (or $d\vec{p}/dt = \vec{F}$).
- Formulate the equation of motion (N2) for m.
- Classify what kind of mathematical equation this is and provide initial or boundary conditions that are needed to solve the equation.
- Solve the equation of motion and answer the two questions.
- Check your math and the result for dimensional correctness. Inspect the limit: $F_{zw} \to 0$.

Exercise 2.103: Acceleration of Gravity 🌶️ 🌶️

- Find an object that you can safely drop from some height.
- Drop the object from any (or several heights) and measure using a stop watch or you mobile the time from dropping to hitting the ground.
- Measure the dropping height.

Find from these data the value of gravity's acceleration constant.

Don't forget to first make an analysis of this experiment in terms of a physical model and make clear what your assumptions are.

Tip

Think about the effect of air resistance: is dropping from a small, a medium or a high height best? Any arguments?

Exercise 2.104: Use numerical analysis to assess influence of air friction 🌶️ 🌶️ 🌶️

If you want to learn also how to use numerical methods ...

Try using an air drag force: $F_{drag} = -A_{\perp} C_D \frac{1}{2} \rho_{air} v^2$. With $A_{\perp}$ the cross-sectional area of your object perpendicular to the velocity vector and $C_D \approx 1$ the drag coefficient (in real life it is actually a function of the velocity). ρ_{air} is the density of air which is about $1.2 kg/m^3$.

Write a computer program (e.g. in python) that calculates the motion of your object. See Solution with Python how you could do that.

Exercise 2.105: Forces on your bike 🌶️

Figure 2.106: Riding a bicycle. Adapted from InjuryMap, from Wikimedia Commons, licensed under CC BY-SA 4.0.

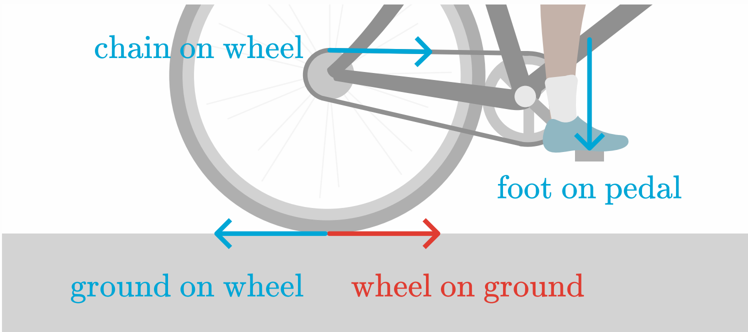
On a bicycle you will have to apply a force to the pedals to move forward, right? What force actually moves you forward, where is it located and who/what is providing that force?

- Make sketch and draw the relevant force. Give the force that actually propels you a different color.
- Think for a minute about the nature of this force: are you surprised?

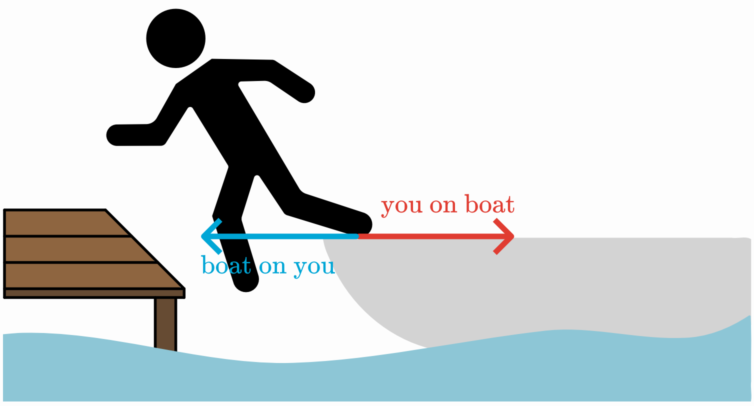
N.B. Consider while thinking about this problem: what would happen if you were biking on an extremely slippery floor?

Solution 2.107: Solution to Exercise 19

When you push with your foot on the pedal, that force is transferred to the chain of your bike. That chain exerts a force on the gear of your bike's rear wheel, trying to get it to rotate. Your wheel touches the ground and, because of the force on the gear, the wheel exerts a force in the ground, trying to push the ground backwards. Due to action=-reaction, the ground exerts a forward force on your wheel. So actually, biking means "making the ground push you forward"!



Exercise 2.109: Getting off the boat 🌶️



You are stepping from a boat onto the shore. Use Newton's laws to describe why you will end up in the water.

N.B. A calculation is not required, but focus on the physics and describe in words why you didn't make it to the jetty.

Solution 2.111: Solution to Exercise 20

When you try to step on the jetty, a force needs to be exerted on you, otherwise you can't move forward. The way you achieve that: you push with your back foot on the boat. And as a result of Newton 3, the boat will push back, but the force from the boat on you is forward directed. That is exactly what you need!

However, while you push, the boat will move backwards due to the force you exert on it. Consequently, your point of contact with the boat shifts away from the jetty. Either you let the boat go and no force from the boat is acting on you. Now gravity will do its work and if your forward velocity is not sufficient, you will not reach the jetty. Or your foot will try to follow the boat and that requires a force to the wrong direction acting on you.

Pushing harder seems an option: your forward velocity might increase more. However, the boat will also be pushed harder and moves quicker away from you. Consequently, the time interval of contact with the boat decreases. Thus, with Newton 2: dp = Fdt your increase in velocity due to the larger force might be compensated by a smaller duration that the force can do so. And you may still end up in the water.

Exercise 2.112: Newton's Laws 🌶

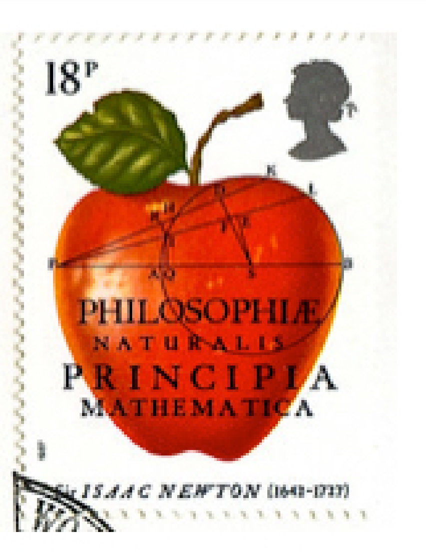


Figure 2.113: align: center

Stamp designs © Royal Mail Group Ltd[^1].

Close this book (or don't peak at it ;-)) and write down Newton's laws. Explain in words the meaning of each of the laws. Try to come up with several, different ways of describing what is in these equations.

2.3 Work and Energy

Updated: 04 feb 2026

2.3.1 Work

Work and energy are two important concepts. Work is the transfer of energy that occurs when a force is applied to an object and causes displacement in the direction of that force, calculated as 'force times path' . However, we need a formal definition:

if a point particle moves from $\vec{r}$ to $\vec{r} + d\vec{r}$ and during this interval a force $\vec{F}$ acts on the particle, then this force has performed an amount of work equal to:

$$dW = \vec{F} \cdot d\vec{r} \tag{2.164}$$

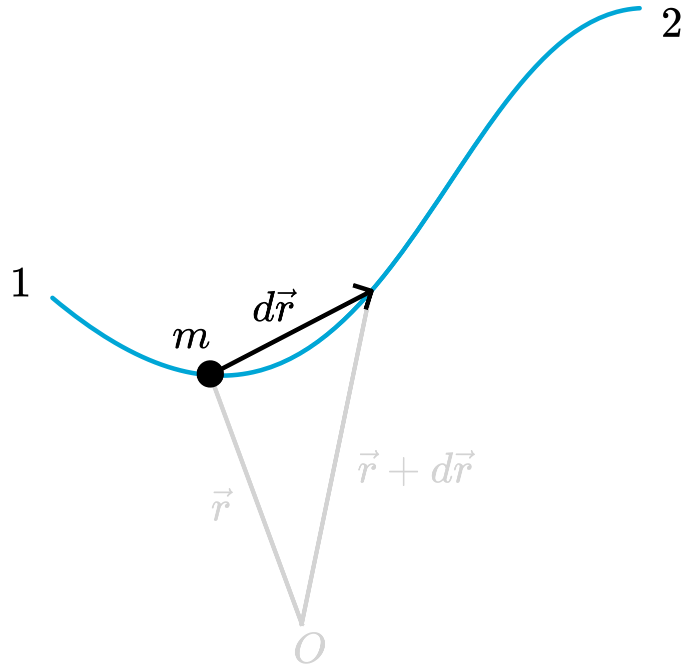


Figure 2.114: Path of a particle.

Note that this an *inner product* between two vectors, resulting in a *scalar* . In other words, work is a number, not a vector. It has no direction. That is one of the advantages over force.

$$dW = \vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz \tag{2.165}$$

Work done on m by F during motion from 1 to 2 over a prescribed trajectory:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} \tag{2.166}$$

Keep in mind: in general the work depends on the starting point 1, the end point 2 and on the trajectory. Different trajectories from 1 to 2 may lead to different amounts of work.

Tip

See also the chapter in the linear algebra book on the inner product

2.3.2 Kinetic Energy

Kinetic energy is defined and derived using the definition of work and Newton's 2^{nd} Law.

The following holds: if work is done on a particle, then its kinetic energy must change. And vice versa: if the kinetic energy of an object changes, then work must have been done on that particle. The following derivation shows this.

Exercise 2.115: Carrying a weight

You carry a heavy backpack $m = 20$ kg from Delft to Rotterdam (20 km). What is the work that you have done against the gravitational force?

$$\begin{aligned} W_{12} &= \int_1^2 \vec{F}\cdot d\vec{r} = \int_1^2 \vec{F}\cdot\frac{d\vec{r}}{dt}dt = \int_1^2 \vec{F}\cdot\vec{v}dt \\ &= \int_1^2 m\frac{d\vec{v}}{dt}\cdot\vec{v}dt = m\int_1^2 \vec{v}\cdot d\vec{v} = m\left[\frac{1}{2}\vec{v}^2\right]_1^2 \\ &= \frac{1}{2}m\vec{v}_2^2 - \frac{1}{2}m\vec{v}_1^2 \end{aligned} \tag{2.167}$$

It is from the above that we indicate $\frac{1}{2}m\vec{v}^2$ as kinetic energy. It is important to realize that the concept of kinetic energy does not bring anything that is not contained in N2 to the table. But it does give a new perspective: kinetic energy can only be gained or lost if a force performs work on the particle. And vice versa: if a force performs work on a particle, the particle will change its kinetic energy.

Obviously, if more than one force acts, the net work done on the particle determines the change in kinetic energy. It is perfectly possible that force 1 adds an amount W to the particle, whereas at the same time force 2 will take out an amount $-W$. This is the case for a particle that moves under the influence of two forces that cancel each other: $\vec{F}_1 = -\vec{F}_2$. From Newton 2, we immediately infer that if the two forces cancel each other, then the particle will move with a constant velocity. Hence, its kinetic energy stays constant. This is in line with the fact that in this case the net work done on the particle is zero:

$$W_1 + W_2 = \int_1^2 \vec{F}_1\cdot d\vec{r} + \int_1^2 \vec{F}_2\cdot d\vec{r} = \int_1^2 \vec{F}_1\cdot d\vec{r} - \int_1^2 \vec{F}_1\cdot d\vec{r} = 0 \tag{2.168}$$

2.3.3 Worked Examples

Reminder of path/line integral from Analysis

As long as the path can be split along coordinate axis the separation above is a good recipe. If that is not the case, then we need to turn back to the way how things have been introduced in the Analysis class. We need to make a 1D parameterization of the path.

Line integral of a vector valued function $\vec{F}(x,y,z):\mathbb{R}^3\to\mathbb{R}^3$ over a curve `cal` C is given as

$$\int_{\texttt{cal}\ C} \vec{F}(x,y,z)\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(\tau))\cdot\frac{d\vec{r}(\tau)}{d\tau}\,d\tau \tag{2.174}$$

We integrate in the definition of the work from point 1 to 2 over an implicitly given path. To compute this actually, you need to parameterize the path by $\vec{r}(\tau) = (x(\tau), y(\tau), z(\tau))$. The integration variable τ tells you where you are on the path, $\tau\in[a,b]\in\mathbb{R}$. The derivative of $\vec{r}$ with respect to τ gives the tangent vector to the curve, the "speed" of walking along the curve. In the analysis class you used $\vec{v}(\tau)\equiv\frac{d\vec{r}(\tau)}{d\tau}$ for the speed. The value of the line integral is independent of the chosen parameterization. However, it changes sign when reversing the integration boundaries.

Example: Another path

Solution 2.116: Solution to Exercise 1

The answer is, of course, zero! That is because the path (from Delft to Rotterdam) is perpendicular to the gravitational force. Therefore the inner product $\vec{F}_g \cdot d\vec{r} = 0$ over the whole way. Let us look at it more formally, this will help us when things get more complicated later.

The force is $\vec{F}_g(x, y, z) = (0, 0, -mg) = -mg\hat{z}$ and we choose our coordinate system such that the path be along the x-axis, the y-coordinate is zero and we the backpack is at height $z = 1$ m.

$$W_g = \int_{Delft}^{Rott} F_x dx + F_y dy + F_z dz = \int F_x dx \,|_{y=0,z=1} = \int 0\, dx = 0 \quad (2.169)$$

So gravity has not performed work on your backpack. Similarly, you have exercised a force $\vec{F}_N$ on the backpack. As the backpack doesn't change its vertical coordinate, we know $\vec{F}_N + \vec{F}_g = 0$. And immediately, we see:

$$W_N = \int_{Delft}^{Rott} F_{Nx} dx + F_{Ny} dy + F_{Nz} dz = \int F_x dx \,|_{y=0,z=1} = \int 0\, dx = 0 \quad (2.170)$$

You didn't perform any work either. This may feel strange or even wrong. After all, you will probably be pretty tired after the walk. However, that is due to the internal working of our muscles and body. In order to sustain the force $\vec{F}_N$ humans do use energy: work is done in their muscles. But from a physics point of view: no work is done on the backpack.

Now we integrate from $(0, 0) \to (1, 1)$ but along the diagonal. A parameterization of this path is $\vec{r}(\tau) = (0, 0) + (1, 1)\tau = (\tau, \tau), \tau \in [0, 1]$. The derivative is $\frac{d\vec{r}(\tau)}{d\tau} = (1, 1)$. Therefore we can write the work of $\vec{F}(x, y) = -y\hat{x} + x^2\hat{y}$ along the diagonal as

$$\begin{gathered}\int_0^1 \vec{F}(\tau, \tau) \cdot (1, 1)\, d\tau = \int_0^1 (-\tau, \tau^2) \cdot (1, 1)\, d\tau = \\ \int_0^1 (-\tau + \tau^2)\, d\tau = -\frac{1}{6}\end{gathered} \quad (2.175)$$

Integration of the same force $\vec{F}(x, y) = -y\hat{x} + x^2\hat{y}$ from $(0, 0) \to (1, 1)$ but along a normal parabola. A parameterization of the path is $\vec{r}(\tau) = (0, 0) + (\tau, \tau^2), \tau \in [0, 1]$ and the derivative is $\frac{d\vec{r}}{d\tau} = (1, 2\tau)$. The work then is

Exercise 2.117: Compressing a spring[6]

You're compressing an uncompressed spring with spring constant k over a distance x. How much work do you need to do?

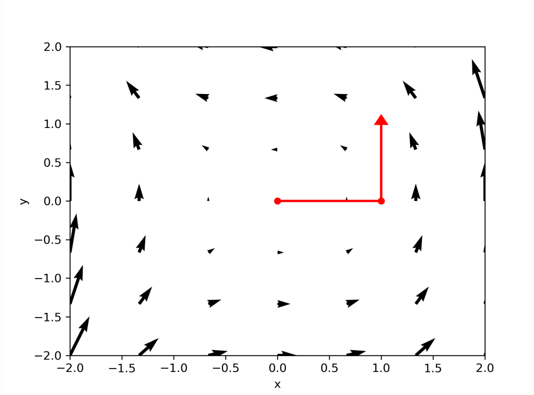
Solution 2.118: Solution to Exercise 2

$$W = \int_{x_1}^{x_2} F \mathrm{d}x = \int_0^x kx \mathrm{d}x = \frac{1}{2}kx^2 \quad (2.171)$$

[6]Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Exercise 2.119: Work in a force field

Now we consider a force field $\vec{F}(x, y) = (-y, x^2) = -y\hat{x} + x^2\hat{y}$. We compute the work done over a path from the origin $(0,0)$ to $(1,0)$ and then to $(1,1)$ first along the x-axis and then parallel to the y-axis.



$$\begin{aligned}
\int_0^1 \vec{F}(\tau, \tau^2) \cdot (1, 2\tau)\, d\tau = \\
\int_0^1 (-\tau^2, \tau^2) \cdot (1, 2\tau)\, d\tau = \\
\int_0^1 (-\tau^2 + 2\tau^3)\, d\tau = \frac{1}{6}
\end{aligned} \tag{2.176}$$

2.3.4 Gravitational potential energy

Let's consider an object close to the surface of any planet, where the acceleration due to gravity can be described by $F_g = -mg$. Raising the object to a height H requires us to do

Solution 2.121: Solution to Exercise 3

We can split up the integral in these two parts as the direction in both parts is constant, therefore the inner product can be separated out.

$$\begin{aligned}
W &= \int_{(0,0)}^{(1,0)} \vec{F} \cdot d\vec{r} + \int_{(1,0)}^{(1,1)} \vec{F} \cdot d\vec{r} \\
&= \int_{(0,0)}^{(1,0)} F_x dx\, |_{y=0} + \int_{(1,0)}^{(1,1)} F_y dy\, |_{x=1} \\
&= \int_{(x=0)}^{(x=1)} -y\, dx\, |_{y=0} + \int_{(y=0)}^{(y=1)} x^2\, dy\, |_{x=1} \\
&= -yx\, |_{x=0}^{x=1}\, |_{y=0} + x^2 y\, |_{y=0}^{y=1}\, |_{x=1} = 1
\end{aligned} \tag{2.172}$$

Try to integrate the force field yourself along a different path $(0,0) \to (0,1) \to (1,1)$ to the same end point.

$$\begin{aligned}
W &= \int_{y=0}^{y=1} F_y\, dy\, |_{x=0} + \int_{x=0}^{x=1} F_x\, dx\, |_{y=1} \\
&= \int_{y=0}^{y=1} x^2\, dy\, |_{x=0} + \int_{x=0}^{x=1} -y\, dx\, |_{y=1} \\
&= -1 + 0 = -1
\end{aligned} \tag{2.173}$$

The work done is not the same over this path. This is already obvious from the graph showing the path and the force field: the second path clearly moves against the force, where the first is moving with direction of the force.

Exercise 2.122: Potential & kinetic energy

Proof that the velocity of an object released from a height H will reach the velocity $v = \sqrt{2gH}$.

work: we will have to apply a force $F = +mg$ to the object to lift it to position H. Thus, with two forces acting - each doing work on the object we get:

$$\begin{aligned} W_g &= \int_0^H F_g dx = \int_0^H -mgdx = -mgH \\ W_+ &= \int_0^H -F_g dx = \int_0^H mgdx = mgH \end{aligned} \tag{2.177}$$

The net effect is of course $W_{net} = 0$ as the object started without kinetic energy and ends without kinetic energy, thus we knew in advance $0 = \Delta E_{kin} = W_g + W_+$

We can also take a slightly different view on this. Suppose we only concentrate on the work done by gravity: $W_g = -mgH$. Note that there is a minus sign, the gravitational force works in the opposite direction of the movement of the object. As energy is a conservative quantity, someone or something has supplied the object with some 'gained' energy. We call this potential energy, more particular in this case gravitational potential energy.

Why is it called 'potential'? When the object is released from that height H, this gravitational potential energy is converted to kinetic energy. The gravitational force does work on the object:

$$W = \int_H^0 F dx = \int_H^0 mgdx = mgH = \Delta E_{kin} \tag{2.178}$$

From this, it follow that the object will reach a velocity of $v = \sqrt{2gH}$. This is an example of a situation where an object looses potential energy and gains kinetic energy.

```
interactive(children=(FloatSlider(value=9.81, description='g (m/s²)',
max=15.0, min=1.5), IntSlider(value=1, d…
```

Exercise 2.123:

A point particle of mass $m = 1$kg is at $t = 0$ at position $x = 0$. It has initial velocity v_0. From $t = 0$ to $t_{stop} = 2$s it is under the influence of a constant force F. This is a 1D problem.

The top graph show the position of the particle. The bottom graph shows the Work done on the particle by the force and the kinetic energy of the particle.

Analyse this situation and calculate the work done by the force at any time. Is the work done in this case always sufficient to account for the change in kinetic energy? What does it mean if the work is positive or negative?

Exercise 2.124:

Use the Python app below, and answer the following questions:

- does the acceleration double when the mass of the falling box doubles?
- the position time diagram is made using kinematics, how would the code look like when based on energy conservation?
- how would you include friction in the code?

Exercise 2.125:

Look at the following roller coaster app.

Change the various graph settings (what is on the x/y axis). Change the starting position of the ball, and try to change the path.

Can you make sense of the motion and the graphs?

```
<function __main__.run_animation(g=9.81, M=1)>
```

2.3.5 Conservative force

As we saw, work done on m by F during motion from 1 to 2 over a prescribed trajectory, is defined as:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} \tag{2.179}$$

In general, the amount of work depends on the path followed. That is, the work done when going from $\vec{r}_1$ to $\vec{r}_2$ over the red path in the figure below, will be different when going from $\vec{r}_1$ to $\vec{r}_2$ over the blue path. Work depends on the specific trajectory followed.

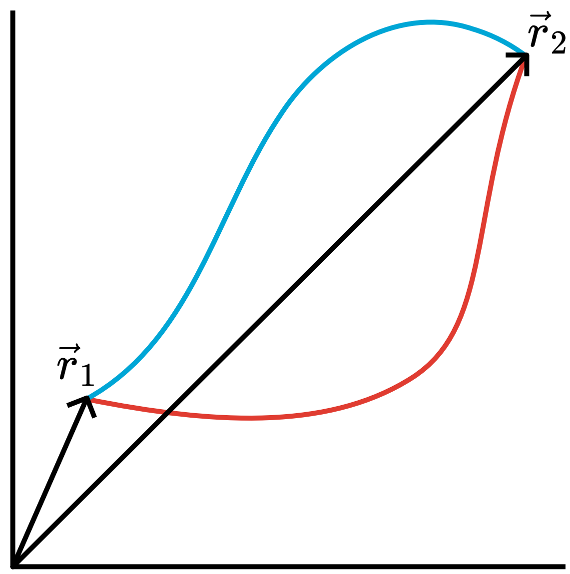


Figure 2.126: Two different paths.

However, there is a certain class of forces for which the path does not matter, only the start and end point do. These forces are called conservative forces. As a consequence, the work done by a conservative force over a closed path, i.e start and end are the same, is always zero. No matter which closed path is taken.

$$conservative force \Leftrightarrow \oint \vec{F} \cdot d\vec{r} = 0 for ALL closed paths \tag{2.180}$$

2.3.5.1 Stokes' Theorem

It was George Stokes who proved an important theorem, that we will use to turn the concept of conservative forces into a new and important concept.

Figure 2.127: Sir George Stokes (1819-1903). From Wikimedia Commons, public domain.

His theorem reads as:

$$\oint \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} \tag{2.181}$$

In words: the integral of the force over a closed path equals the surface integral of the curl of that force. The surface being ‘cut out’ by the close path. The term $\vec{\nabla} \times \vec{F}$ is called the curl of F:, which is a vector. The meaning of the curl and some words on the theorem are given below.

Intermezzo: intuitive proof of Stokes’ Theorem

Consider a closed curve in the -plane. We would like to calculate the work done when going around this curve. In other words: what is if we move along this curve?We can visualize what we need to do: we cut the curve in small part; compute for each part (i.e. the red, green, blue, etc. in and sum these to get the total along the curve. If we make the parts infinitesimally small, we go from a (Riemann) sum to an integral.Closed path on a grid.However, we are going to compute much more: take a look at . We have put a grid in the -plane over a closed curve . Hence, the interior of our curve is full of squares. We are not only computing the parts along the curve, but also along the sides of alle curves. This will sound like way too much work, but we will see that it actually is a very good idea.See : we calculate counter clockwise for the green square. Than we have at least the green part of our done in the right direction. Hence, we compute along the right side of the green square. We do that from bottom to top as we go counter clockwise along the green square. Let’s call that .Then we move to the blue square and repeat in counter clockwise direction our calculation. But this means that we compute along the left side of blue the square from top to bottom. We will call this .Next, we will add all contributions. Thus we get . But these two cancel each other as they are exactly the same but done in opposite directions. Thus if we use that for any integration, it becomes obvious that .Note that this will happen for all side of the squares that are in the interior of our curve. Thus, the integral over all squares is exactly the integral along the curve .It seems, we do a lot of work for nothing. But there is another way of looking at the path-integrals along the squares. If we make the square small enough, the calculation along one square can be approximated:

Example: Work done in a vectorfield

Suppose we need to calculate the integral of the vectorfield $\vec{F}(x, y) = y\hat{x} - x\hat{y}$ over the closed curve formed by a square from $(0, 0)$ to $(1, 0)$, $(1, 1)$, $(0, 1)$ and back to $(0, 0)$.

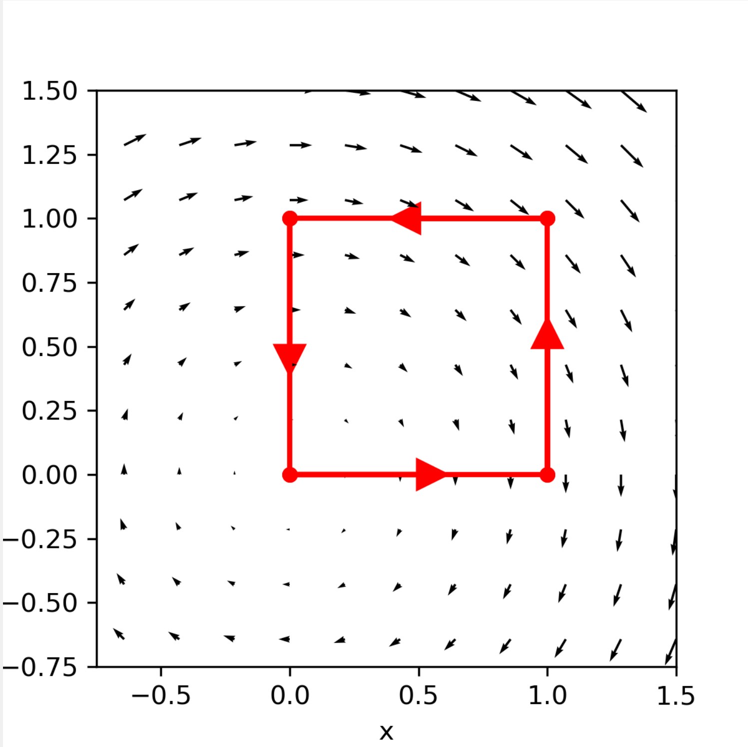


Figure 2.128: Integrating along the unit square.

We go counter clockwise.

$$\begin{aligned}
\oint \vec{F} \cdot d\vec{r} &= \int_{x=0}^{1} F_x(x, y=0)dx + \int_{y=0}^{1} F_y(x=1, y)dy + \\
&\quad + \int_{x=1}^{0} F_x(x, y=1)dx + \int_{y=1}^{0} F_y(x=0, y)dy \\
&= \int_{0}^{10} dx + \int_{0}^{1} -1dy + \int_{1}^{01} dx + \int_{1}^{0} -0dx \\
&= 0 - [y]_0^1 + [x]_1^0 - 0 \\
&= -2
\end{aligned} \tag{2.182}$$

Now we try this using Stokes' Theorem:

$$\oint \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} \tag{2.183}$$

We first calculate $\vec{\nabla} \times \vec{F}$:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \left(\frac{\partial(-x)}{\partial x} - \frac{\partial(y)}{\partial y} \right)\hat{z} = -2\hat{z} \tag{2.184}$$

Thus, in this example $\vec{\nabla} \times \vec{F}$ has only a z-component.

An elementary surface element of the square is: $d\vec{\sigma} = dxdy\hat{z}$. This also has only a z-component. Note that it points in the positive z-direction. This is a consequence of the counter clockwise direction that we use to go along the square.

According to Stokes Theorem, we this find:

$$\oint \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} = \int_{x=0}^{1} \int_{y=0}^{1} (-2) dx dy = -2 \tag{2.185}$$

Indeed, we find the same outcome.

2.3.5.2 Conservative force and $\vec{\nabla} \times \vec{F}$

For a conservative force the integral over the closed path is zero for any closed path. Consequently, $\vec{\nabla} \times \vec{F} = 0$ everywhere. How do we know this? Suppose $\vec{\nabla} \times \vec{F} \neq 0$ at some point in space. Then, since we deal with continuous differentiable vector fields, in the close vicinity of this point, it must also be non-zero. Without loss of generality, we can assume that in that region $\vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} > 0$. Next, we draw a closed curve around this point, in this region. We now calculate the $\oint \vec{F} \cdot d\vec{r}$ along this curve. That is, we invoke Stokes Theorem. But we know that $\vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} > 0$ on the surface formed by the closed curve. Consequently, the outcome of the surface integral is non-zero. But that is a contradiction as we started with a conservative force and thus the integral should have been zero.
The only way out, is that $\vec{\nabla} \times \vec{F} = 0$ everywhere.

Thus we have:

$$conservative force \Leftrightarrow \vec{\nabla} \times \vec{F} = 0 everywhere \tag{2.186}$$

2.3.6 Potential Energy

This function V is called the potential energy or the potential for short and has a direct connection to the work. A direct consequence of the above is:

if $\vec{\nabla} \times \vec{F} = 0$ everywhere, a function $V(\vec{r})$ exists such that $\vec{F} = -\vec{\nabla} V$

$$\begin{gathered} conservative force \Leftrightarrow \vec{\nabla} \times \vec{F} = 0 everywhere \\ \Updownarrow \\ \vec{F} = -\vec{\nabla} V \Leftrightarrow V(\vec{r}) = -\int_{ref} \vec{F} \cdot d\vec{r} \end{gathered} \tag{2.187}$$

where in the last integral, the lower limit is taken from some, self picked, reference point. The upper limit is the position $\vec{r}$.

Next to its direct connection to work, the potential is also connected to kinetic energy.

$$E_{kin,2} - E_{kin,1} = W_{12} = \int_{1}^{2} \vec{F} \cdot d\vec{r} = V(\vec{r}_2) - V(\vec{r}_1) \tag{2.188}$$

or rewritten:

$$E_{kin,1} + V(\vec{r}_1) = E_{kin,2} + V(\vec{r}_2) \tag{2.189}$$

In words: for a conservative force, the sum of kinetic and potential energy stays constant.

2.3.6.1 Energy versus Newton's Second Law

We, starting from Newton's Laws, arrived at an energy formulation for physical problems. Question: can we also go back? That is: suppose we would start with formulating the energy rule for a physical problem, can we then back out the equation of motion?
Answer: yes, we can!

It goes as follows. Take a system that can be completely described by its kinetic plus potential energy. Then: take the time-derivative and simplify, we will do it for a 1-dimensional case first.

$$
\begin{aligned}
&\frac{1}{2}mv^2 + V(x) = E_0 \Rightarrow \\
&\frac{d}{dt}\left[\frac{1}{2}mv^2 + V(x)\right] = \frac{dE_0}{dt} = 0 \Rightarrow \\
&mv\dot{v} + \frac{dV}{dx}\underbrace{\frac{dx}{dt}}_{=v} = 0 \Rightarrow \\
&v\left(m\dot{v} + \frac{dV}{dx}\right) = 0
\end{aligned}
\tag{2.190}
$$

The last equation must hold for all times and all circumstances. Thus, the term in brackets must be zero.

$$
m\dot{v} + \frac{dV}{dx} = 0 \Rightarrow m\ddot{x} = -\frac{dV}{dx} = F \tag{2.191}
$$

And we have recovered Newton's second law.

In 3 dimensions it is the same procedure. What is a bit more complicated, is using the chain rule. In the above 1-d case we used $\frac{dV}{dt} = \frac{dV(x(t))}{dt} = \frac{dV}{dx}\frac{dx(t)}{dt}$. In 3-d this becomes:

$$
\frac{dV}{dt} = \frac{dV(\vec{r}(t))}{dt} = \frac{dV}{d\vec{r}} \cdot \frac{d\vec{r}(t)}{dt} = \vec{\nabla}V \cdot \vec{v} \tag{2.192}
$$

Thus, if we repeat the derivation, we find:

$$
\begin{aligned}
&\frac{1}{2}mv^2 + V(\vec{r}) = E_0 \Rightarrow \\
&\frac{d}{dt}\left[\frac{1}{2}mv^2 + V(\vec{r})\right] = 0 \Rightarrow \\
&m\vec{v} \cdot \dot{\vec{v}} + \vec{\nabla}V \cdot \vec{v} = 0 \Rightarrow \\
&v\left(m\vec{a} + \vec{\nabla}V\right) = 0 \Rightarrow \\
&m\vec{a} = -\vec{\nabla}V = \vec{F}
\end{aligned}
\tag{2.193}
$$

And we have recovered the 3-dimensional form of Newton's second Law. This is a great result. It allows us to pick what we like: formulate a problem in terms of forces and momentum, i.e. Newton's second law, or reason from energy considerations. It doesn't matter: they are equivalent. It is a matter of taste, a matter of what do you see first, understand best, find easiest to start with. Up to you!

2.3.7 Stable and Unstable Equilibrium

A particle (or system) is in equilibrium when the sum of forces acting on it is zero. Then, it will keep the same velocity, and we can easily find an inertial system in which the particle is at rest, at an equilibrium position.
The equilibrium position (or more general: state) can also be found directly from the potential energy.

Potential energy and (conservative) forces are coupled via:

$$
\vec{F} = -\vec{\nabla}V \tag{2.194}
$$

The equilibrium positions $\left(\sum_i \vec{F}_i = 0\right)$ can be found by finding the extremes of the potential energy:

$$
equilibriumposition \Leftrightarrow \vec{\nabla}V = 0 \tag{2.195}
$$

Once we find the equilibrium points, we can also quickly address their nature: is it a stable or unstable solution? That follows directly from inspecting the characteristics of the potential energy around the equilibrium points.

For a stable equilibrium, we require that a small push or a slight displacement will result in a force pushing back such that the equilibrium position is restored (apart from the inertia of the object that might cause an overshoot or oscillation).

However, an unstable equilibrium is one for which the slightest push or displacement will result in motion away from the equilibrium position.

The second derivative of the potential can be investigated to find the type of extremum. For 1D functions that is easy, for scalar valued functions of more variables that is a bit more complicated. Here we only look at the 1D case $V(x) : \mathbb{R} \to \mathbb{R}$

$$equilibrium : \vec{\nabla} V = 0 \tag{2.196}$$

Luckily, the definition of potential energy is such that these rules are easy to visualize in 1D and to remember, see Figure 7

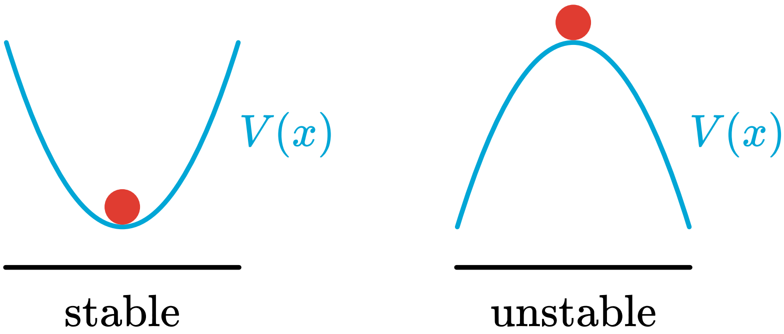


Figure 2.129: Stable and unstable position of a particle in a potential.

A valley is stable; a hill top is unstable.
NB: Now the choice of the minus sign in the definition of the potential is clear. Otherwise a hill would be stable, but that does not feel natural at all.

It is also easy to visualize what will happen if we distort that particle from the equilibrium state:

- The valley, i.e., the stable system, will make the particle move back to the lowest point. Due to inertia, it will not stop but will continue to move. As the lowest position is one of zero force, the particle will 'climb' toward the other end of the valley and start an oscillatory motion.
- The top, i.e., the unstable point, will make the particle move away from the stable point. The force acting on the particle is now pushing it outwards, 'down the slope of the hill'.

2.3.7.1 Taylor Series Expansion of the Potential

The Taylor expansion or Taylor series is a different way of writing down the value of a function in the vicinity of a point x_0. Even though the function is written down in a different way, it is equal to f in the vicinity of x_0. It uses an infinite series of polynomial terms with coefficients given by value of the derivative of the function at that specific point x_0. The value of the terms for higher n become small, so we can approximate the function by using only the first few terms. The more of these first terms you take, the closer your approximation is. Mathematically, it reads for a 1D scalar function $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) \approx f(x_0) + \frac{1}{1!}f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f'''(x_0)(x - x_0)^3 + \tag{2.197}$$

For our purpose here, it suffices to stop after the second derivative term:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \mathcal{O}(x^3) \tag{2.198}$$

A way of understanding why the Taylor series actually works is the following.
Imagine you have to explain to someone how a function looks around some point x_0, but

you are not allowed to draw it. One way of passing on information about $f(x)$ is to start by giving the value of $f(x)$ at the point x_0:

$$f(x) \approx f(x_0) \tag{2.199}$$

Next, you give how the tangent at x_0 is: you pass on the first derivative at x_0. The other person can now see a bit better how the function changes when moving away from x_0:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \tag{2.200}$$

Then, you tell that the function is not a straight line but curved, and you give the second derivative. So now the other one can see how it deviates from a straight line:

$$f(x) \approx f(x_0) + \frac{1}{1!}f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 \tag{2.201}$$

Note that the prefactor is placed back. But the function is not necessarily a parabola; it will start deviating more and more as we move away from x_0. Hence we need to correct that by invoking the third derivative that tells us how fast this deviation is. And this process can continue on and on.

Important to note: if we stay close enough to x_0 the terms with the lowest order terms will always prevail as higher powers of $(x - x_0)$ tend to zero faster than a lower powers (for instance: $0.5^4 << 0.5^2$).

This 3Blue1Brown clip explains the 1D Taylor series nicely.

Figure 2.130: A 3blue1brown clip on Taylor series.

For scalar valued functions as our potentials $V(\vec{r}) : \mathbb{R}^3 \to \mathbb{R}$ the extension of the Taylor series is not too difficult. If we expand the function around a point

$$\begin{aligned} V(\vec{r}) \approx\ & V(\vec{r}_0) + \vec{\nabla}V(\vec{r}_0) \cdot (\vec{r} - \vec{r}_0) \\ & + \frac{1}{2}(\vec{r} - \vec{r}_0) \cdot (\partial^2 V)(\vec{r}_0) \cdot (\vec{r} - \vec{r}_0) + \mathcal{O}(r^3) \end{aligned} \tag{2.202}$$

The second derivative of the potential indicated by $\partial^2 V$ is the Hessian matrix. Right now, this all sound a bit hocus pocus. But don't worry: you won't need it right away in its full glory. In the rest of your physics and math classes, this will all come back and start to make sense.

Conceptually the extrema of the function are again the hills and valleys. The classification of the extrema has next to hills and valleys also saddle points etc. In this course we will not bother about these more dimensional cases, but only stick to simple ones.

2.3.8 Examples, exercises and solutions

Updated: 04 feb 2026

2.3.8.1 Worked Examples

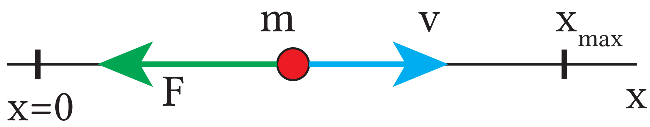
2.3.8.1.1 Slowing a mass down

We continue with the worked example from chapter 2: a mass, m, has initial velocity v_0. From $t = 0$ onwards a force F (with magnitude μmg, $\mu > 0$) is acting on m, slowing it down. Eventually, m will come to a stand still. The problem is 1-dimensional.

How much work has F done?

Interpret the problem

First we make a sketch and draw what is relevant for this problem. We can use the same figure as used in chapter 2.



We conclude our interpret-phase with our idea on how to approach this problem:

- the force is a friction force, hence it is not conservative.
- work and change of kinetic energy are related: $W_{12} = E_{kin,2} - E_{kin,1}$
- we do know the velocity of m at the beginning and at the end: v_0 and 0, respectively.

Develop the solution

We can use the relation between work and kinetic energy:

$$W_{12} = E_{kin,2} - E_{kin,1} \tag{2.203}$$

We know the right hand side: $E_{kin,1} = \frac{1}{2}mv_0^2$ and $E_{kin,2} = 0$

Alternatively, we could also compute the work directly from its definition. In a 1-dimensional case that is:

$$W_{12} \equiv \int_1^2 F dx \tag{2.204}$$

Note: F has sign; it is not the magnitude of F.

Evaluate the solution

From the relation between work and kinetic energy we get that the work done is:

$$W_{12} = 0 - \frac{1}{2}mv_0^2 = -\frac{1}{2}mv_0^2 \tag{2.205}$$

The alternative approach using $W_{12} = \int_1^2 F dx$ can be solved by realizing: F points in the negative x-direction, i.e. $F = -\mu mg$. It is a constant along the path from 1 to 2, hence integration is simple.

$$W_{12} = \int_0^{x_{max}} -\mu mg dx = -\mu mg x_{max} \tag{2.206}$$

We have now two answers and, obviously, the should be the same.

Assess the solution

From the first approach we learn: $W_{12} < 0$. That makes sense as the particle looses kinetic energy: it is slowed down by F.

The same conclusion is drawn from the second approach. After all, $x_{max} > 0$.

To convince ourselves that we the two answers are the same, we need to go back to the solution of the problem in chapter 2. There we found that $x_{max} = \frac{1}{2}\frac{v_0^2}{\mu g}$.

If we substitute this in our answer using the direct integration of F over the path, we get:

$$W_{12} = -\mu m g x_{max} = -\mu m g \frac{1}{2}\frac{v_0^2}{\mu g} = -\frac{1}{2} m v_0^2 \tag{2.207}$$

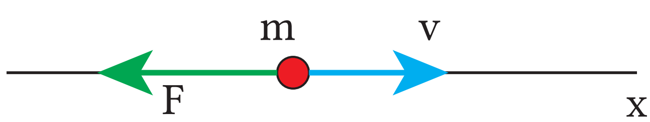
As expected: we get the same answer.

2.3.8.1.2 Is friction a conservative force?

As a second example: let us investigate of the friction force, $F = -\mu m g$, of the above example a conservative force is or not.

Interpret the problem

Again, we start with a sketch and draw what is relevant for this problem. We can use a similar figure as above, but with even less detail.



As we deal with a 1-dimensional problem, vector arrows above F, v or x are not needed. But don't be deceived: when dealing with work we have to evaluate an inner product and even in a 1-dimensional case quantities like force have direction.

We can proceed via different directions:

1. find a potential or show that this doesn't exist;
2. show that $\oint \vec{F} \cdot d\vec{r} = 0$ always or find at least one closed path for which the work is non-zero.

We opt for the second approach: find one path from $x = x_1$ to $x = x_1$ for which $\int_{x_1}^{x_1} F dx \neq 0$.

Develop and Evaluate the solution

Before finding such a path, let's for a moment think about a potential $V(x)$. It seems almost too simple:

$$V(x) \equiv -\int_{x_{ref}}^{x} F(x')dx' = \mu m g \int_0^x dx' = \mu m g x \tag{2.208}$$

where we have chosen: $x_{ref} = 0$.

So: do we conclude F is conservative? After all, we seem to have found a potential! And if we use this, we quickly find the point where m comes to a stand still:

$$E_{kin} + V(x) = E_0 \Rightarrow \frac{1}{2}mv_0^2 + \mu mg0 = \\ = \frac{1}{2}m0^2 + \mu mgx_{max} \rightarrow x_{max} = \frac{1}{2}\frac{v_0^2}{\mu g} \tag{2.209}$$

which is the correct answer.

However: our answer is right, our reasoning is wrong.

How to understand this? Let's go back to the closed path. If F is conservative: the work done on m when it goes from $x = 0$ to x_{max} most be the exact opposite from te work done when m moves from x_{max} to $x = 0$. So, let us compute this.

$$W_{0 \rightarrow x_{max}} = \int_0^{x_{max}} \mu mgdx' = \mu mgx_{max} \tag{2.210}$$

$$W_{x_{max} \rightarrow 0} = \int_{x_{max}}^0 -\mu mgdx' = \int_0^{x_{max}} \mu mgdx' = \mu mgx_{max} \tag{2.211}$$

It is exactly equal! Of course, this is due to the nature of the friction force: it flips sign if m moves in the other direction.

Assess the solution

We conclude: $W_{0 \rightarrow x} + W_{x \rightarrow 0} \neq 0$ and our force is not conservative; $V(x)$ does not exist!

Where did the calculation of $V(x)$ go wrong?

The integration seems quite ok, and it is. The problem is that we are sloppy with hoe we write F. We use $F = -\mu mg$ if m moves in the positive x-direction and $F = \mu mg$ if m moves in the negative x-direction. Actually: F depends on the sign of the velocity v. We should have written something like: $F = -sgn(v)\ \mu mg$ with $sgn(v) = 1$ if $v > 0$, $= -1$ if $v < 0$ and $= 0$ if $v = 0$.

Our lesson: if *sloppy notation* then *errors are around the corner*.

2.3.8.1.3 Cycling in a force field

The professor likes to cycle in a force field during his break. The force field is given by: $F = y\hat{x}$. As he has to return for his next lecture, he cycles in a closed loop. He moves from (0,0) to (1,0) to (1,1) to (0,1) to (0,0) - or the other way around. Given that he has to have enough energy to educate the students, he wonders: how much work do I do during the ride and does that differ when I go clockwise?

Interpret the problem

Let's first make a sketch of the situation. In this case we need to get some idea of the force field and the path. We can do that by plotting some vectors. This can be done by hand, but also by using Python:

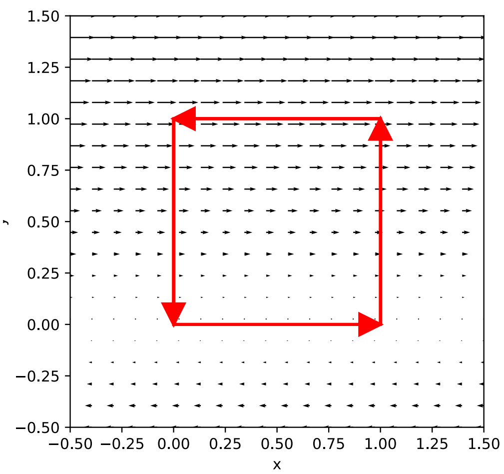


Figure 2.133: The force field that the professor is cycling in.

Develop the solution

Looking at the force field and the path, we can see clearly that the path matters: from (1,1) to (0,1) the force is pointing in the opposite direction of the movement. At all other instances we can see that the force is either zero, or orthogonal to the movement.

Hence, to find the work done we only need to consider the segment from (1,1) to (0,1) and find $W = \int F \cdot ds$

Evaluate the solution

The only segment that contributes to the work done is the one from (1,1) to (0,1). Along this segment we have $F = 1\hat{x}$ as exerted by the force field, hence $F = -1\hat{x}$ exerted by the professor, and $ds = -1\hat{x}$, or:

$$W = \int_1^0 F \cdot ds = \int_1^0 -1 \cdot ds = 1\text{J} \tag{2.212}$$

Assess the solution

Although this is a simple case where drawing both the force field and the path helps a lot, we could have checked mathematically whether the force field is conservative or not. As described in conservative forces, a force field is conservative if the curl of the force field ($\vec{\nabla} \times \vec{F}$) is zero. In our case this means:

$$\begin{aligned}\vec{\nabla} \times \vec{F} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} \\ &= \left(\frac{\partial(0)}{\partial y} - \frac{\partial(0)}{\partial z}\right)\hat{x} + \left(\frac{\partial(0)}{\partial x} - \frac{\partial(y)}{\partial z}\right)\hat{y} + \left(\frac{\partial(0)}{\partial x} - \frac{\partial(y)}{\partial y}\right)\hat{z} \\ &= -1\hat{z} \neq 0\end{aligned} \tag{2.213}$$

Hence, the force field is not conservative and the work done depends on the path taken. This is in line with our earlier conclusion.

Exercise 2.134: Gravity, a conservative force? 🌶️

Is gravity $\vec{F}_g = m\vec{g}$ a conservative force? If yes, what is the corresponding potential energy?

To find the answer:

a. Show $\vec{\nabla} \times m\vec{g} = 0$
b. Find a V that satisfies $-m\vec{g} = -\vec{\nabla}V$

Exercise 2.135: 🌶️

A point particle of mass $m = 1$ kg is at $t = 0$ at position $x = 0$. It has initial velocity v_0. From $t = 0$ to $t_{stop} = 2$ s it is under the influence of a constant force F. This is a 1D problem.

The top graph shows the position of the particle. The bottom graph shows the Work done on the particle by the force and the kinetic energy of the particle.

Analyze this situation and calculate the work done by the force at any time. Is the work done in this case always sufficient to account for the change in kinetic energy? What does it mean if the work is positive or negative?

Exercise 2.136: 🌶️

A simple model for the frictional force experienced by a body sliding over a horizontal, smooth surface is $F_f = -\mu F_g$ with F_g the gravitational force on the object. The friction force is opposite the direction of motion of the object.

- Show that this frictional force is not conservative (and, consequently, a potential energy associated does not exist!).

Tip

Think of two different trajectories to go from point 1 to point 2 and show that the amount of work along these trajectories is not the same.

Or: find a closed loop for which the work done by the frictional force is non-zero.

Exercise 2.137: 🌶️

A force is given by: $\vec{F} = x\hat{x} + y\hat{y} + z\hat{z}$

- Show that this force is conservative.
- Find the corresponding potential energy.

A second force is given by: $\vec{F} = y\hat{x} + x\hat{y} + z\hat{z}$

- Show that this force is also conservative.
- Find the corresponding potential energy.

Exercise 2.138: 🌶️

Another force is given by: $\vec{F} = y\hat{x} - x\hat{y}$

- Show that this force is not conservative.
- Compute the work done when moving an object over the unit circle in the xy-plane in an anti-clockwise direction. (Hint: use Stokes theorem.)
- Discuss the meaning of your answer: is it positive or negative? And what does that mean in terms of physics?

Exercise 2.139: 🌶️

Given a potential energy $E_{pot} = xy$.
a. Find the corresponding force (field).
b. Make a plot of $\vec{F}$ as a function of (x,y,z).
c. Describe the force and comment on what the potential itself already reveals about the force.

Exercise 2.140: 🌶️

Given a force field $\vec{F} = -xy\hat{x} + xy\hat{y}$. A particle moves from $(x, y) = (0, 0)$ over the x-axis to $(x, y) = (1, 0)$ and then parallel to the y-axis to $(x, y) = (1, 1)$. In a second motion, the same particle goes from $(x, y) = (0, 0)$ over the y-axis to $(x, y) = (0, 1)$ and then parallel to the x-axis to end also in $(x, y) = (1, 1)$.

- Show that not necessarily the work done over the two paths is equal.
- Compute the amount of work done over each of the paths.

Exercise 2.141: 🌶️ 🌶️

A particle of mass m is initially at position $x = 0$. It has zero velocity. On the particle a force is acting. The force can be described by $F = F_0 \sin \frac{x}{L}$ with F_0 and L positive constants.

1. Show that this force is conservative and find the corresponding potential. Take as reference point for the potential energy $x = \frac{\pi}{2}L$.
2. The particle gets a tiny push, such that it starts moving in the positive x-direction. Its initial velocity is so small that, for all practical calculations, it can be set to zero. What will happen to the particle after the push?
3. Find the maximum velocity that the particle can get. At which location(s) will this take place?

Note: this is a 1-dimensional problem.

Solution 2.142: Solution to Exercise 1

a. Show $\vec{\nabla} \times m\vec{g} = 0$
$\vec{\nabla} \times m\vec{g} = 0$? How to compute it? For **Cartesian** coordinates there is an easy to remember rule:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{2.214}$$

If we chose our coordinates such that $\vec{g} = -g\hat{z}$ we get:

$$\vec{\nabla} \times \vec{F}_g = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -mg \end{vmatrix} = 0 \tag{2.215}$$

Thus $\vec{F}_g$ is conservative.

b. Find a V that satisfies $-m\vec{g} = -\vec{\nabla}V$
Does $-m\vec{g} = -\vec{\nabla}V$ have a solution for V? Let's try, using the same coordinates as above.

$$\begin{aligned} -\vec{\nabla}V &= -m\vec{g} \Rightarrow \\ \frac{\partial V}{\partial x} &= 0 \rightarrow V(x,y,z) = f(y,z) \\ \frac{\partial V}{\partial y} &= 0 \rightarrow V(x,y,z) = g(x,z) \\ \frac{\partial V}{\partial z} &= mg \rightarrow V(x,y,z) = mgz + h(x,y) \end{aligned} \tag{2.216}$$

f,g,h are unknown functions. But all we need to do, is find one V that satisfies $-m\vec{g} = -\vec{\nabla}V$.

So, if we take $V(x,y,z) = mgz$ we have shown, that gravity in this form is conservative and that we can take $V(x,y,z) = mgz$ for its corresponding potential energy.

By the way: from the first part (curl F = 0), we know that the force is conservative and we know that we could try to find V from

$$\begin{aligned} V(x,y,z) &= -\int_{ref} m\vec{g} \cdot d\vec{r} = \int_{ref} mg\hat{z} \cdot d\vec{r} \\ &= \int_{ref} mgdz = mgz + const \end{aligned} \tag{2.217}$$

Solution 2.143: Solution to Exercise 3

Click for the solution Friction Not Conservative.

Solution 2.144: Solution to Exercise 4

Click for the solution Conservative Force.

Solution 2.145: Solution to Exercise 5

Click for the solution Non-Conservative Force.

Solution 2.146: Solution to Exercise 6

Click for the solution Potential energy & Force.

Solution 2.147: Solution to Exercise 7

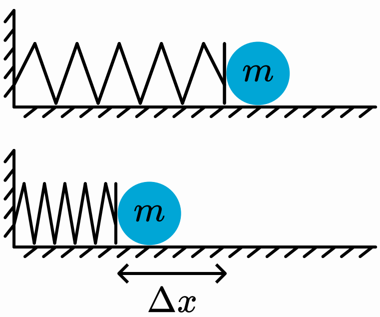
Click for the solution Force Field.

Solution 2.148: Solution to Exercise 8

Click for the solution Sinusoidal Force Field.

Exercise 2.149: Shooting a ball using a spring 🌶️

A ball with mass m is horizontally pressed against a spring with spring constant k, compressing the spring by Δx.



1. Express the velocity of the ball when released.
2. Why is in real life the actual velocity of the ball less (friction is not the answer)?
3. Why is the velocity of the ball less when shot vertically?

Exercise 2.151: Firing a cannon ball[7] 🌶️

1. Show that, if you ignore drag, a projectile fired at an initial velocity v_0 and angle θ has a range R given by

$$R = \frac{v_0^2 \sin 2\theta}{g} \tag{2.218}$$

1. A target is situated 1.5 km away from a cannon across a flat field. Will the target be hit if the firing angle is $42°$ and the cannonball is fired at an initial velocity of 121 m/s? (Cannonballs, as you know, do not bounce).
2. To increase the cannon's range, you put it on a tower of height h_0. Find the maximum range in this case, as a function of the firing angle and velocity, assuming the land around is still flat.

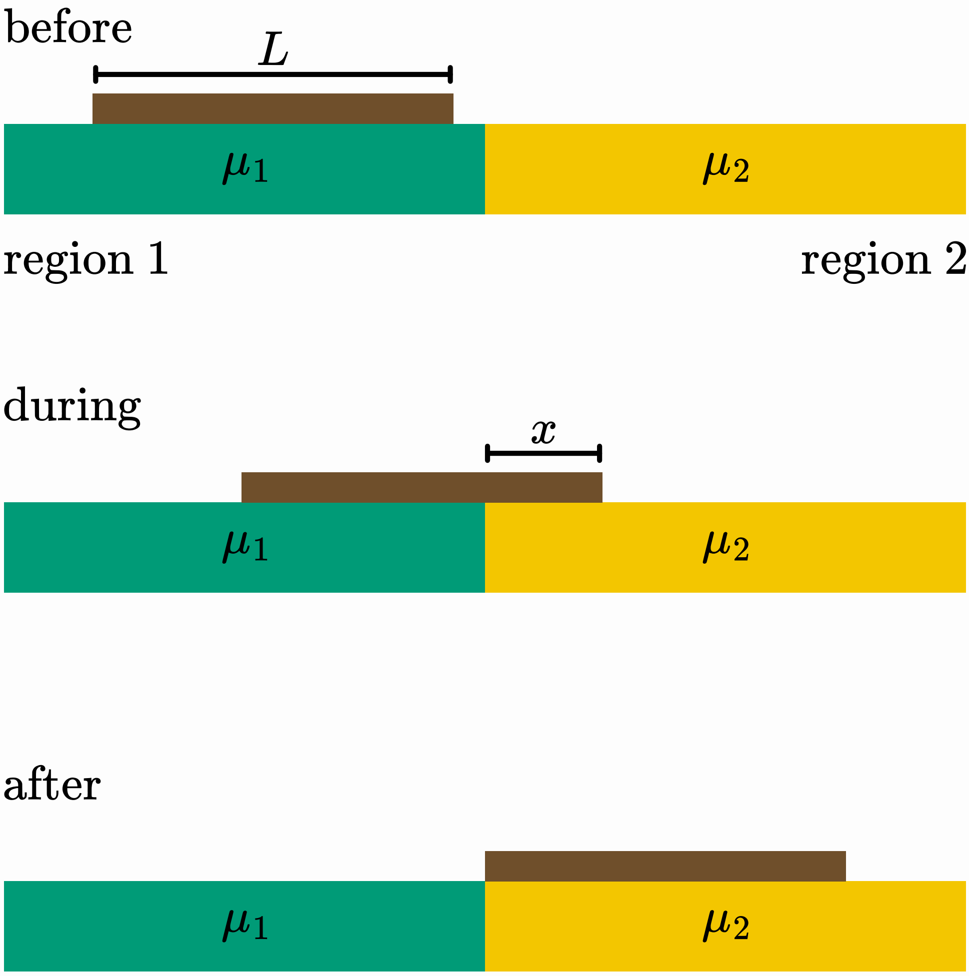
Exercise 2.152: Pushing a box uphill[8] 🌶️

You push a box of mass m up a slope with angle θ and kinetic friction coefficient μ. Find the minimum initial speed v you must give the box so that it reaches a height h. Use energy and work to find the answer.

[7] Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)
[8] Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Exercise 2.153: Work done dragging a board[9] 🌶️ 🌶️

A uniform board of length L and mass M lies near a boundary that separates two regions. In region 1, the coefficient of kinetic friction between the board and the surface is μ_1, and in region 2, the coefficient is μ_2. Our objective is to find the net work W done by friction in pulling the board directly from region 1 to region 2, under the assumption that the board moves at constant velocity.



1. Suppose that at some point during the process, the right edge of the board is a distance x from the boundary, as shown. When the board is at this position, what is the magnitude of the force of friction acting on the board, assuming that it's moving to the right? Express your answer in terms of all relevant variables (L, M, g, x, μ_1, and μ_2).
2. As we have seen, when the force is not constant, you can determine the work by integrating the force over the displacement, $W = \int F(x)\mathrm{d}x$. Integrate your answer from (1) to get the net work you need to do to pull the board from region 1 to region 2.

[9] Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Exercise 2.155: 🌶️ 🌶️

A point particle (mass m) drops from a height H downwards. It starts with zero initial velocity. The only force acting is gravity (take gravity's acceleration as a constant).

- Set up the equation of motion (i.e. N2) for m.
- Calculate the velocity upon impact with the ground.
- Calculate the kinetic energy of m when it hits the ground.
- Calculate the amount of work done by gravity by solving the integral $W_{12} = \int_1^2 \vec{F} \cdot d\vec{r}$.
- Show that the amount of work calculated is indeed equal to the change in kinetic energy.

Solve this exercise using IDEA.

Exercise 2.156: 🌶️ 🌶️

A hockey puck ($m = 160$ gram) is hit and slides over the ice-floor. It starts at an initial velocity of 10m/s. The hockey puck experiences a frictional force from the ice that can be modeled as $-\mu F_g$ with F_g the gravitational force on the puck. The friction force eventually stops the motion of the puck. Then the friction is zero (otherwise it would accelerate the puck from rest to some velocity :smiley:). Constant $\mu = 0.01$.

- Set up the equation of motion (i.e. N2) for m.
- Solve the equation of motion and find the trajectory of the puck.
- Calculate the amount of work done by gravity by solving the integral $W_{12} = \int_1^2 \vec{F} \cdot d\vec{r}$.
- Show that the amount of work calculated is indeed equal to the change in kinetic energy.
- Solve this exercise using IDEA.

Exercise 2.157: 🌶️

An electron (mass m, charge -e) is in a static electric field. The electric field is of the form $\vec{E} = E_0 \sin\left(2\pi \frac{X}{L}\right)\hat{x}$. As a consequence, the electron experiences a force $\vec{F} = -e\vec{E}$ Due to this force, the electron moves from position $x = \frac{1}{4}L$ to $x = 0$.

- Calculate the amount of work done by the electric field.
- Assuming that the electron was initially at rest, what is the velocity at $x = 0$?

Exercise 2.158:

A force $F = F_0 e^{-t/\tau}$ is acting on a particle of mass m. The particle is initially at position $x = 0$. It is, starting at $t = 0$, moving at a constant velocity $v_0 > 0$ to $x = L$, $(L > 0)$.

a. Calculate the amount of work done by F.
b. The amount of work done is equal to the change in kinetic energy. However, the particle doesn't change its kinetic energy. Why is this general rule not violated in this case?

Exercise 2.159: Work by a lineair force

A point particle of mass $m = 2$kg is at $t = 0$ at position $x = 0$. It has initial velocity v_0. From $t = 0$ to $t_{stop} = 4$s it is under the influence of a force $F(x)$ that linearly increases with the position: $F(x) = kx$ with $k > 0$. This is a 1D problem.

The top graph show the position of the particle. The bottom graph shows the Work done on the particle by the force and the kinetic energy of the particle.

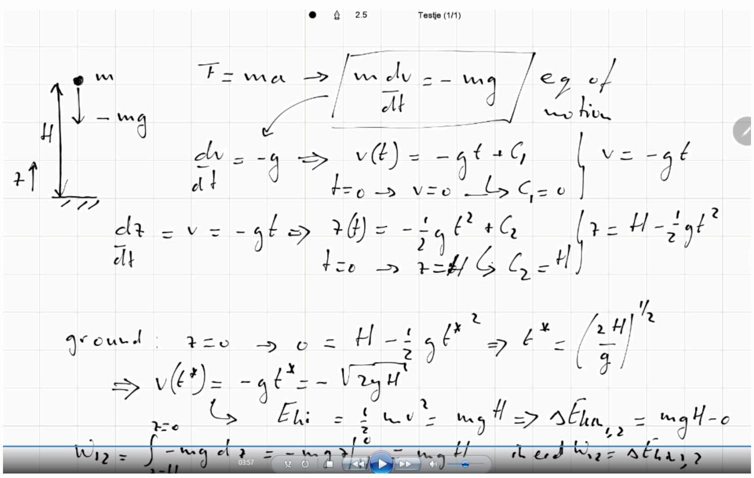
Analyse this situation and calculate the work done by the force at any time. Is the work done in this case always sufficient to account for the change in kinetic energy? What does it mean if the work is positive or negative?

Are the red (W) line and the green (E_{kin}) parallel? What does that mean?

Solution 2.160: Solution to Exercise 9

1. $W = \Delta E_{kin} = \int_0^x F\mathrm{d}x = \int_0^x kx dx = 1/2kx^2 = 1/2mv^2 \Rightarrow v = \sqrt{\frac{kx^2}{m}}$
2. The spring has mass as well.
3. The gravitational does work as well ($W = F_g \mathrm{d}x < 0$)

Solution 2.161: Solution to Exercise 13



Solution 2.163: Solution to Exercise 15

Work done by electric field when the electron moves from $x = \frac{1}{4}L$ to $x = 0$:

$$W = \int_{\frac{1}{4}L}^{0} \vec{F} \ \ .d\vec{s} = -eE_0 \int_{\frac{1}{4}L}^{0} \sin\Big(2\pi \frac{x}{L}\Big) dx = -eE_0 \frac{L}{2\pi} \Big[-\cos\Big(2\pi \frac{x}{L}\Big)\Big]_{\frac{1}{4}L}^{0} = \frac{1}{2\pi} eE_0 L \tag{2.219}$$

Work done is gain in kinetic energy: $\Delta E_{kin} = W$. Assuming the only work done is by the electric field and using initial velocity is zero: $v_i = 0$:

$$\frac{1}{2}mv^2 = \frac{1}{2\pi} eE_0 L \Rightarrow v = \sqrt{\frac{eE_0 L}{\pi m}} \tag{2.220}$$

Note that indeed the work done is positive, as it should in this case since the electron starts with zero velocity.

2.3.8.2 Exercise set 1

2.3.8.3 Answers set 1

2.3.8.4 Exercise set 2

2.3.8.5 Answers set 2

Solution 2.164: Solution to Exercise 16

$$W = \int_0^L \vec{F} \ \ .d\vec{s} = \int_0^L F_0 e^{-\frac{t}{\tau}} \ \ dx \tag{2.221}$$

Particle velocity is $v_0 = const$. Thus, trajectory $x(t) = v_0 t$ since at $t = 0 \rightarrow x = 0$ Consequently: $x = L \rightarrow t = \ \frac{L}{v_0}$

Thus, we can write for the amount of work done:

$$W = \int_0^{\frac{L}{v_0}} F_0 e^{-\frac{t}{\tau}} \cdot v_0 dt = F_0 v_0 \Big[-\tau e^{-\frac{t}{\tau}}\Big]_0^{L/v_0} = F_0 v_0 \tau \Big(1 - e^{-\frac{L}{v_0 \tau}}\Big) \tag{2.222}$$

We note: $W > 0$ and naively, we could expect that the kinetic energy of the particle would have increased. But that isn't the case: it started with $E_{kin} = \frac{1}{2}mv_0^2$ and it kept this along the entire path as it is given that the particle is traveling with a constant velocity.

From this last statement, we immediately learn, that there must be a second force acting on the particle. This force is exactly equal and opposite to F at all times! Otherwise, the particle would accelerate and change its velocity. Consequently, this second force also perform work on m, the amount is exactly $-W$ and thus the total work done on the particle is zero which reflects that the particle does not change its kinetic energy.

2.4 Angular Momentum, Torque & Central Forces

Updated: 04 feb 2026

2.4.1 Torque & Angular Momentum

From experience we know that if we want to unscrew a bottle, lift a heavy object on one side, try to unscrew a screw, we better use 'leverage'.

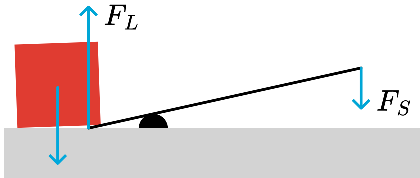


Figure 2.165: Lifting is easier using leverage.

With a relatively small force F_S, we can lift the side of a heavy object. The key concept to use here is torque, which in words is loosely formulated: apply the force using a long arm and the force seems to be magnified. The torque is then force multiplied by arm:

$\tau = Force \ \ times \ \ arm$

This is, of course, too sloppy for physicists. We need strict, formal definitions. So, we put the above into a mathematical definition.

torque

$$\vec{\tau} \equiv \vec{r} \times \vec{F} \tag{2.223}$$

That is: torque (or krachtmoment in Dutch) is the cross product of 'arm' as a vector(!) and the force. We notice a few peculiarities.

1. like force, torque is a vector. That is: it has a magnitude and a direction. In principle: three components.
2. its direction is perpendicular to the force vector $\vec{F}$ *and* perpendicular to the arm $\vec{r}$.
3. the arm is not a number: it is a vector!

We further know from experience that we can balance torques, like we can balance forces. Rephrased: the net effect of more than one force is found by adding all the forces (as vectors!) and using the net force in Newtons Second Law: $m\vec{a} = \sum \vec{F}_i = \vec{F}_{net}$. From Newtons First Law, we immediately infer: if $\sum \vec{F}_i = \vec{F}_{net} = 0$ then the object moves at constant velocity. We can move with the object at this speed and conclude that it from this perspective has zero velocity: it doesn't move, i.e. it is in equilibrium.

The same holds for torque: we can work with the sum of all torques that act on an object: $\sum \vec{\tau}_i = \vec{\tau}_{net}$. And if this sum is zero, the object is in equilibrium.

However, there is a catch: using torques requires that we are much more explicit and precise about the choice of our origin. Why? The reason is in the 'arm'. That is only defined if we provide an origin.

2.4.1.1 The seesaw and torque

Let's consider a simple example (simple in the sense that we are all familiar with it): the seesaw.

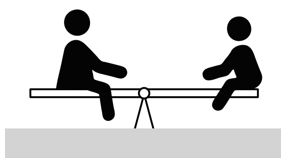


Figure 2.166: An adult (left) and a child (right) on a seesaw.

It is obvious that the adult -seesawing with the child- should sit much closer to the pivot point than the child. That is: we assume that the mass of the adult is greater than that of the child.

Let's turn this picture into one that captures the essence and includes the necessary physical quantities, and then draw a free-body diagram.

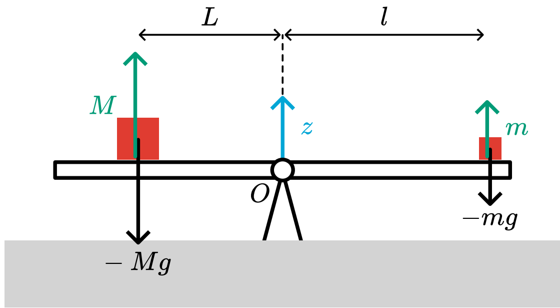


Figure 2.167: Free-body diagram of the seesaw and the masses.

What did we **draw**?

1. The force of gravity acting on the two masses M and m. That is obvious: without forces nothing will happen and there is nothing to be analyzed.
2. The 'reaction forces' from the seesaw on both masses. Why? If the seesaw is in equilibrium, then each of the masses is in equilibrium and the sum of forces on each mass must be zero.
3. The distance of each of the masses to the pivot point. Why? Leverage! The heavy M must be closer to the pivot point the get equilibrium.
4. An origin O. Why? We need a point to measure the 'arm', 'leverage', from.
5. The z-coordinate. Why? We deal with forces in the vertical direction. Hence a coordinate, a direction that we all use, is handy.

Analysis
Time for a first analysis: what keeps this seesaw in equilibrium?

1. The sum of forces on each of the masses is zero. As gravity pulls them down, the seesaw must exert a force of the same magnitude but in the opposite direction. These are the green forces.
2. With this idea we have the masses in equilibrium, but not necessarily the seesaw. Why? We did not consider forces on the seesaw. Which are these: (a) the reaction force (i.e. the N3 pair) of the green force from the seesaw on mass M. We did not draw that! Similarly, for the mass m.
3. Now that we focus on the seesaw itself: this is in equilibrium (that is given), but there are two forces acting on it in the negative z-direction as we found in (2). Even if we consider the mass of the seesaw: that will not help, gravity will pull it downwards. What did we forget? The force at the pivot point, of course! The pivot will exert an upward force, preventing the seesaw from falling down. For simplicity, we assume that the seesaw has zero mass. Thus, there are three forces acting on it: $-Mg$, $-mg$, F_p with $F_p - Mg - mg = 0$.

Let's redraw, now concentrating on the forces on the seesaw.

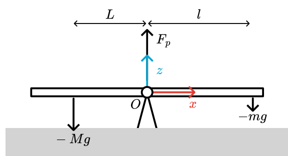


Figure 2.168: Free-body diagram of the seesaw.

Analysis part 2
We know that the seesaw is in equilibrium, thus

$$F_p - Mg - mg = 0 \tag{2.224}$$

This guarantees that the seesaw does not change its velocity, and as it does not move at some time t_0, it doesn't move for all $t > t_0$.

But this doesn't guarantee that the seesaw doesn't rotate around the pivot point. For that we need that the 'leverages' on the left and right side 'perform' the same.
Making this precise: the torques exerted on the seesaw must also equate to zero.
We have 3 forces, thus 3 torques: $-Mg$ with arm L, $-mg$ with arm l and F_p with arm zero.

Now we need to be even more precise: torque is a vector and it is made as a cross product of the vector 'arm' and the force.
We have already drawn an x-coordinate in the figure, that will allow us to write the 'arm' as a vector. After all, we need to evaluate the cross product $\vec{r} \times \vec{F}$. We do that for the three forces, starting on the left:

$$\vec{\tau}_1 = -L\hat{x} \times (-Mg)\hat{z} = MLg\,\hat{x} \times \hat{z} = MLg(-\hat{y}) = -MLg\,\hat{y} \tag{2.225}$$

We have used here, that the cross product of $\hat{x}$ with $\hat{z}$ is equal to $-\hat{y}$ with $\hat{y}$ the unit vector in the y-direction pointing into the screen.

Similarly for the force coming from the small mass m on the right side:

$$\vec{\tau}_2 = l\hat{x} \times (-mg)\hat{z} = -mlg\,\hat{x} \times \hat{z} = mlg\,\hat{y} \tag{2.226}$$

Finally, the torque from the force exerted by the pivot point:

$$\vec{\tau}_3 = 0\hat{x} \times F_p\hat{z} = 0 \tag{2.227}$$

Next, we evaluate the total torque:

$$\vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 = (mlg - MLg)\hat{y} \tag{2.228}$$

In order for the seesaw not to start rotating, we must have that the torque is zero and thus:

$$\sum \vec{\tau}_i = 0 \Rightarrow mlg = MLg \rightarrow \frac{m}{M} = \frac{L}{l} \tag{2.229}$$

A result we expected: the greater mass should be closer to the pivot point.

2.4.1.2 Different origin

So far, so good. But what if we had chosen the origin not at the pivot point, but somewhere to the left? Then all 'arm' will change length. And all torques will be different. Wouldn't that make $\sum \vec{\tau}_i \neq 0$?
No, it wouldn't! Let's just do it and recalculate. In the figure below, we have moved the origin to the left end of the seesaw. The distance from the heavy mass to the origin is Λ (green arrow).

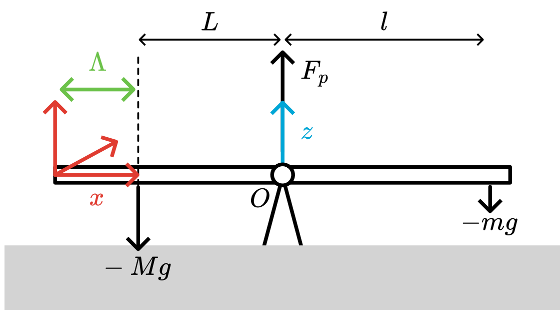


Figure 2.169: Free-body diagram with the origin located at the seesaw's end.

We still have that the sum of forces is zero. But what about the sum of torques? Obviously, the choice of the origin cannot affect the seesaw: it is still in balance, regardless of our choice of the origin. Let's see if that is correct:

$$\sum \vec{\tau}_i = \Lambda\hat{x} \times -Mg\hat{z} + (\Lambda + L) \times F_p\hat{z} + (\Lambda + L + l)\hat{x} \times -mg\hat{z} \tag{2.230}$$

We have drawn the three unit vectors $\hat{x}, \hat{y}, \hat{z}$ in the figure. We will use again: $\hat{x} \times \hat{z} = -\hat{y}$. This simplifies the torque equation above to:

$$\sum \vec{\tau}_i = \left[Mg\Lambda - (\Lambda + L)F_p + mg(\Lambda + L + l)\right]\hat{y} \tag{2.231}$$

This is clearly more complicated than the expression we had with the first choice of the origin. Moreover, the force from the pivot point shows up in our expression.

Luckily, we have equilibrium. Hence: $F_p - Mg - mg = 0 \Rightarrow F_p = Mg + mg$. We substitute this into our torque equation:

$$\begin{aligned}\sum \vec{\tau}_i &= [Mg\Lambda - (\Lambda + L)(Mg + mg)) + mg(\Lambda + L + l)]\hat{y} \\ &= [Mg(\Lambda - (\Lambda + L)) + mg(-(\Lambda + L) + \Lambda + L + l)]\hat{y} \\ &= [-MgL + mgl]\hat{y}\end{aligned} \tag{2.232}$$

Which is exactly the same expression as we found before. So, indeed, the choice of the origin doesn't matter.

Conclusion

For equilibrium we need that the sum of torques is zero:

$$\sum_i \vec{\tau}_i = 0 \tag{2.233}$$

2.4.2 Angular Momentum

From our seesaw example we learn: the seesaw can only be in equilibrium if the sum of torques is zero. What if this sum is non-zero? That is, a net torque operates on the seesaw.

We know that the seesaw will rotate and in order to balance it, we have to shift at least one of the masses.

In which direction will it rotate?

Before answering: first we need to think about **direction of rotation**. Does it have direction and if so: how do we make clear what that is?

Again the seesaw will give guidance. Suppose we remove the smaller mass all together. Then, it is obvious: the 'heavy' left side will rotate to the ground and the light right side upwards. From the point of view we are standing: the seesaw will rotate counter clockwise.

We will use the corkscrew rule or right hand rule to give that a direction: rotate a corkscrew clockwise and the screw will move into the cork away from you; rotate a corkscrew counter clockwise and it will move out of the cork, towards you. Of course, in stead of a corkscrew you can think of a screwdriver or a water tap: closing is rotating 'clock wise, opening the tap is counter clockwise.

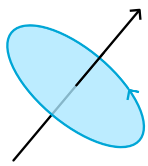


Figure 2.170: The rotation is given by the black arrow. You can find it by using the corkscrew rule: rotating a corkscrew as the blue arrow indicates gives that the screw moves forward like the black arrow.

With this, we can define the direction of rotation better than via clock or counter clock wise. In our seesaw example, we will say: if the seesaw rotates clockwise, its rotation is described by a vector that points in the positive y-direction as given in the figure, that is pointing into the paper (or screen).

Now, we can couple this to the direction of the torque. We saw -taking the origin at the pivot point- two torques acting on the seesaw. The large mass has its torque pointing in the negative y-direction: it points out of the screen/paper. And this torque tries to rotate the seesaw counter clockwise. On the other hand, the small mass has a torque pointing in the positive y-direction which is in line with it trying to rotate the seesaw clockwise.
Which of the two is 'strongest' determines the direction of rotation: if $MgL > mgl$ then the net torque is in the minus-y direction. That is, the torque of the larger mass is more negative than the smaller one is positive: $-MgL + mgl < 0$ and the net torque points towards us.

The quantity that goes with this, is the angular momentum. It is defined as

angular momentum

$$\vec{l} \equiv \vec{r} \times \vec{p} \tag{2.234}$$

Note that it is a cross product as well. Hence it is a vector itself. Further note that $\vec{r} \times \vec{p} \neq \vec{p} \times \vec{r}$. The order matters! First $\vec{r}$ then $\vec{p}$. If you do it the other way around, you unwillingly have introduced a minus sign that should not be there.
Furthermore, note that since $\vec{l} \equiv \vec{r} \times \vec{p}$, $\vec{l}$ is perpendicular to the plane formed by $\vec{r}$ and $\vec{p}$.

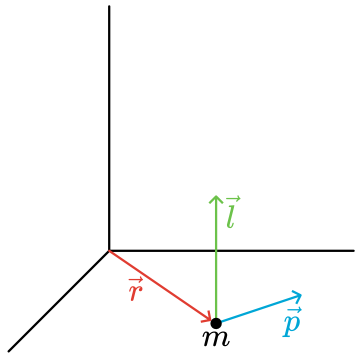


Figure 2.171: Angular momentum of a particle at a certain position with momentum.

2.4.2.1 Torque & Analogy to N2

Angular momentum obeys a variation of Newton's second law that ties it directly to torque.

$$\vec{l} = \vec{r} \times \vec{p} \Rightarrow \tag{2.235}$$

$$\frac{d\vec{l}}{dt} = \frac{d(\vec{r} \times \vec{p})}{dt} = \underbrace{\underbrace{\frac{d\vec{r}}{dt}}_{=\vec{v}} \times \vec{p}}_{=0 since \vec{v} // \vec{p}} + \vec{r} \times \underbrace{\frac{d\vec{p}}{dt}}_{N2:=\vec{F}} = \vec{r} \times \vec{F} \tag{2.236}$$

Thus, we find a general law for the angular momentum:

N2 for angular momentum

$$\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} \tag{2.237}$$

Again, note that the right hand side is a cross product, so the order does matter.

With the torque denoted by $\vec{\tau}$, we have

$$\vec{\tau} \equiv \vec{r} \times \vec{F} \tag{2.238}$$

then we can write down an equation similar to N2 $\left(\dot{\vec{p}} = \vec{F}\right)$ but now for angular motion

$$\dot{\vec{l}} = \vec{\tau} \tag{2.239}$$

where the force is replaced by the torque and the linear momentum by the angular momentum.

NB: Note that the torque and angular moment change if we choose a different origin as this changes the value of $\vec{r}$.

Intermezzo: cross product

Here is some recap for the cross product. See also the Lin. Alg. book page.
A cross product of two vectors and is defined as

It is a common mistake to identify angular momentum with rotational motion. That is not correct. A particle that travels in a straight line will, in general, also have a non-zero angular momentum, see Figure 11. Here we look at a free particle: there are no forces working on it. So it travels in a straight line, with constant momentum.

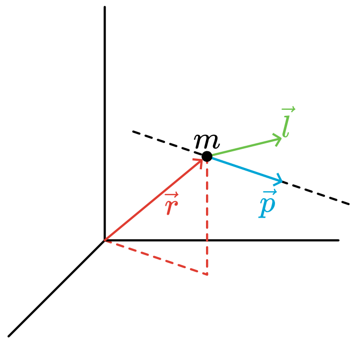


Figure 2.172: Angular momentum of a free particle.

However, the particle position does change over time. So: is its angular momentum constant or not?
That is easy to find out. We could take 'N2' for angular momentum:

$$\frac{d\vec{l}}{dt} = \vec{r} \times \underbrace{\vec{F}}_{\substack{=0 \\ free particle}} = 0 \Rightarrow \vec{l} = const \tag{2.240}$$

Clearly, the angular momentum of a free particle is constant. Moreover, the momentum of a free particle is also constant. But what about the position vector: isn't that changing over time and eventually becomes very, very long? Why does that not change $\vec{r} \times \vec{p}$?
Take a look at Figure 12. We have chosen the xy-plane such that both $\vec{r}$ and $\vec{p}$ are in it. Furthermore, we have taken it such that $\vec{p}$ is parallel to the x-axis.

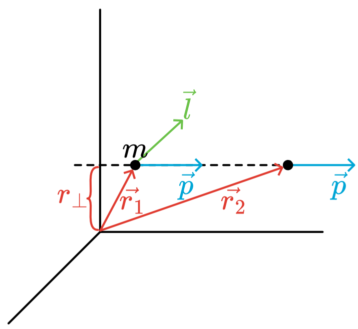


Figure 2.173: Angular momentum of a free particle is constant.

At some point in time, the particle is at position $\vec{r}_1$. Its angular momentum is perpendicular to the xy-plane and has magnitude $|\,|\vec{r}_1 \times \vec{p}\,|\,| = r_\perp p$. Later in time it is at position $\vec{r}_2$. Still, its angular momentum is perpendicular to the xy-plane and has magnitude $|\,|\vec{r}_2 \times \vec{p}\,|\,| = r_\perp p$, indeed identical to the earlier value. This shows that indeed the angular momentum of a free particle is constant.

2.4.3 Examples & Exercises

Example: Throwing a basketball

As seen in class: one person throws a basketball to another via a bounce on the ground, the basketball starts to spin after hitting the ground although initially it did not.

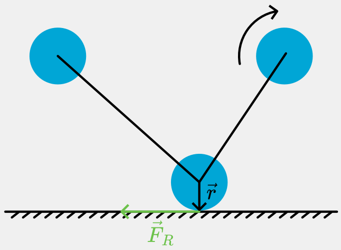


Figure 2.174: A bouncing basketball.

When the ball hits the ground a friction force is acting on the ball. This force will apply a torque on the ball. The friction is directed opposite to the direction of motion. The arm $\vec{r}$ from the center of the ball to where the force is acting, is downwards. Using the right-hand rule we find that the torque is pointing in the plane of the screen, and thus the rotation is clockwise (forwards spin).

The forwards momentum of the ball is reduced by the action of the force. The upwards components is just flipped by the bounce on the ground. Therefore the outgoing ball is bouncing up at a steeper angle than it is was incoming.

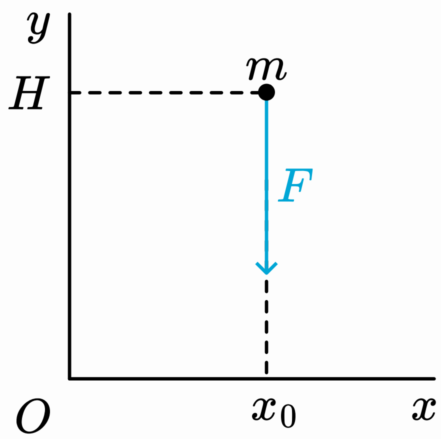
Conservation of angular momentum & spinning wheel

As seen in class, we have a student sitting on a chair that can rotate (swivel chair). The student is holding a bicycle wheel in horizontal position.

Figure 2.175: Student with a rotating wheel on a swivel chair.

Exercise 2.176:

A point particle (mass m) is initially located at position $P = (x_0, H, 0)$. At $t = 0$, it is released from rest and falls in a force field of constant acceleration $\vec{a} = (0, -a, 0)$ that acts on the mass.



Analyze what happens to the angular momentum of m.

Once the student starts to spin the wheel while sitting on the chair, the student will start to rotate in the opposite direction (with smaller angular velocity, later on we will see why their speeds are different). There is no external force on the student + wheel. Consequently, the total angular momentum must stay constant. But the student exerts an angular momentum on the wheel, causing it to rotate. But at the same time, due to action = - reaction, the wheel exerts also a torque on the student. But in the opposite direction. Thus, to compensate the angular momentum pointing up (counter clockwise rotation), an angular momentum pointing down (clockwise rotation) of the same magnitude must occur, keeping the total angular momentum of student + wheel constant.

2.4.4 Central Forces

We have looked at a specific class of forces: the conservative ones. Here we will inspect a second class, that is very useful to identify: the central forces.

A force is called a central force if:

$$\vec{F} = \mid \vec{F}(\vec{r}) \mid \hat{r} \tag{2.250}$$

In words: a force is central if it points always into the direction of the origin or exactly in the opposite direction. The reason to identify this class is in the consequences it has for the angular momentum.

Exercise 2.178:

The same question, but now the particle has an initial velocity $\vec{v} = (v_0, 0, 0)$.

Exercise 2.179:

Similar situation: can you find an example of a falling object for which the angular momentum stays constant? Ignore friction with the air.

Solution 2.180: Solution to Exercise 1

The particle falls under a force that points in the negative y-direction. As a consequence, it will start moving vertically downwards:

$$\begin{aligned} x: h(1cm)m\frac{dv_x}{dt} &= 0 \rightarrow v_x = C_1 = 0 \\ y: h(1cm)m\frac{dv_y}{dt} &= -ma \rightarrow v_y = -at + C_2 = -at \end{aligned} \tag{2.241}$$

Thus, we find for the momentum om the particle: $\vec{p} = (0, -m\,at)$.

The position of m follows from:

$$\begin{aligned} x: h(1cm)\frac{dx}{dt} &= v_x = 0 \rightarrow x(t) = C_3 = x_0 \\ y: h(1cm)\frac{dy}{dt} &= v_y = -at \rightarrow y(t) = -\frac{1}{2}at^2 + C_4 = H - \frac{1}{2}at^2 \end{aligned} \tag{2.242}$$

We can now compute the angular momentum:

$$\begin{aligned} \vec{l} &= \vec{r} \times \vec{p} \\ &= \left(x_0\hat{x} + \left(H - \frac{1}{2}at^2\right)\hat{y}\right) \times (-mat\hat{y}) \\ &= -mx_0at\underbrace{\hat{x} \times \hat{y}}_{=\hat{z}} + x_0\left(H - \frac{1}{2}at^2\right)\underbrace{\hat{y} \times \hat{y}}_{=0} \\ &= -mx_0at\,\hat{z} \end{aligned} \tag{2.243}$$

Thus, the angular momentum is pointing in the negative z-direction and increases linearly with time in magnitude.

We could have tried to find this via the variation of N2 for angular momentum. Now, we need to compute the torque on m and solve $\frac{d\vec{l}}{dt} = \vec{\tau}$. This goes as follows:

$$\begin{aligned} \vec{\tau} &= \vec{r} \times \vec{F} \\ &= (x\hat{x} + y\hat{y}) \times -ma\hat{y} \\ &= -ma\,x\hat{z} \end{aligned} \tag{2.244}$$

And thus:

$$\frac{d\vec{l}}{dt} = -ma\,x\hat{z} \tag{2.245}$$

To get any further, we need information about $x(t)$. From the x-component of N2 we know (see above): $x(t) = x_0$. If we plug this in, we get:

$$\frac{d\vec{l}}{dt} = -ma\,x_0\hat{z} \rightarrow \vec{l} = -mx_0at + C_5 = -mx_0at \tag{2.246}$$

where we have used: $t = 0 \rightarrow \vec{p} = 0 \rightarrow \vec{l} = 0 \Rightarrow C_5 = 0$

Suppose, a particle of mass m is subject to a central force. Then we can immediately infer that its angular momentum is a constant:

$$if\vec{F} = \mid \vec{F}(\vec{r}) \mid \hat{r}then\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} = \mid \vec{F}(\vec{r}) \mid \vec{r} \times \hat{r} = 0 \tag{2.251}$$

where we have used that $\vec{r}$ and $\hat{r}$ are always parallel so their cross-product is zero.

Solution 2.181: Solution to Exercise 2

We can follow the same procedure as in exercise (6.1). But now, the outcome of the x-component of N2 changes.

$$\begin{aligned} x &: h(1cm) m\frac{dv_x}{dt} = 0 \to v_x = C_1 = v_0 \\ y &: h(1cm) m\frac{dv_y}{dt} = -ma \to v_y = -at + C_2 = -at \end{aligned} \tag{2.247}$$

Thus, we find for the momentum om the particle: $\vec{p} = (mv_0, -mat)$.

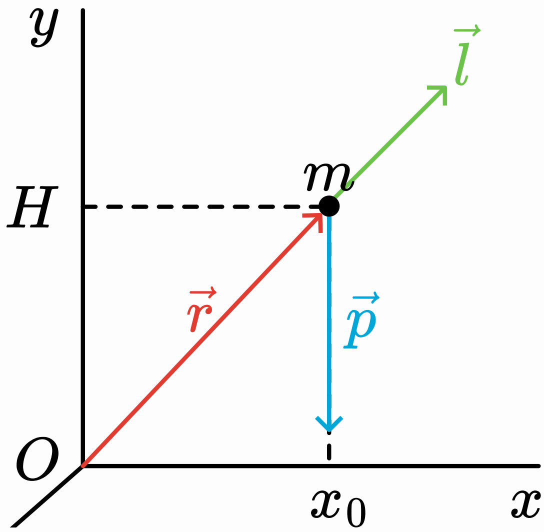
The position of m follows from:

$$\begin{aligned} x &: h(1cm)\frac{dx}{dt} = v_x = v_0 \to x(t) = v_0 t + C_3 = x_0 + v_0 t \\ y &: h(1cm)\frac{dy}{dt} = v_y = -at \to y(t) = -\frac{1}{2}at^2 + C_4 = H - \frac{1}{2}at^2 \end{aligned} \tag{2.248}$$

We can now compute the angular momentum:

$$\begin{aligned} \vec{l} &= \vec{r} \times \vec{p} \\ &= \left((x_0 + v_0 t)\hat{x} + \left(H - \frac{1}{2}at^2\right)\hat{y}\right) \times (mv_0\hat{x} - mat\hat{y}) \\ &= -m(x_0 + v_0 t)at\underbrace{\hat{x} \times \hat{y}}_{=\hat{z}} + \left(H - \frac{1}{2}at^2\right)mv_0\underbrace{\hat{y} \times \hat{x}}_{=-\hat{z}} \\ &= -m\left(Hv_0 + x_0 at + \frac{1}{2}v_0 at^2\right)\hat{z} \end{aligned} \tag{2.249}$$

Thus, the angular momentum still points in the negative z-direction but increases quadratically with time in magnitude.



The above is rather trivial, but has a very important consequence: a particle that moves under the influence of a central force, moves with a constant angular momentum (vector!) and must move in a plane. It cannot get out of that plane. Thus its motion is at maximum a 2-dimensional problem. We can always use a coordinate system, such that the motion of the particle is confined to only two of the three coordinates, e.g. we can choose our x, y plane such that the particle moves in it and thus always has $z(t) = 0$.

Why is this so? Why does the fact that the angular momentum vector is a constant immediately imply that the particle motion is in a plane? The argumentation goes as

follows.
Imagine a particle that moves under the influence of a central force. At some point in time it will have position $\vec{r}_0$ and momentum $\vec{p}_0$. Neither of them is zero. We will assume that $\vec{r}_0$ and $\vec{p}_0$ are not parallel (in general they will not be). Thus they define a plane. Due to the cross-product $\vec{l}_0 = \vec{r}_0 \times \vec{p}_0$ is perpendicular to this plane.
A little time later, say Δt later, both position and momentum will have changed. Since the force is central, the force is also in the plane defined by the initial position and momentum. Thus the change of momentum is in that plane as well: $\vec{p}(t + \Delta t) = \vec{p}(t) + \vec{F}\Delta t$. The right hand side is completely in our plane. And thus, the new momentum is also in the plane. But that means that the velocity is also in the same plane. An thus the new position $\vec{r}(t + \Delta t) = \vec{r}(t) + \frac{\vec{p}}{m}\Delta t$ must be in the same plane as well. We can repeat this argument for the next time and thus see, that both momentum and position cannot get out of the plane. This is, of course, fully in agreement with the fact that $\vec{l} = const$ for a central force.

2.4.5 Central forces: conservative or not?

We can further restrict our class of central forces:

$$if \vec{F}(\vec{r}) = f(r)\hat{r} then F is central and conservative \tag{2.252}$$

In the above, $\mid \vec{F}(\vec{r}) \mid = f(r)$, that is: *the magnitude of the force only depends on the distance from the origin not on the direction.* **Rephrased**: *the force is spherically symmetric.* If that is the case, the force is automatically conservative and a potential does exist.

Both the concept of central forces and potential energy play a pivotal role in understanding the motion of celestial bodies, like our earth revolving the sun. The planetary motion is an example of using the concept of central forces. It is, however, also an example in its own right. Using his new theory, Newton was able to prove that the motion of the earth around the sun is an ellipsoidal one. It helped changing the way we viewed the world from geo-centric to helio-centric.

2.4.5.1 Keppler's Laws

Before we embark at the problem of the earth moving under the influence of the sun's gravity, we will go back in time a little bit.

Intermezzo: Tycho Brahe & Johannes Kepler

We find ourselves back in the Late Renaissance, that is around 1550-1600 AD. In Europe, the first signs of the scientific revolution can be found. Copernicus proposed his heliocentric view of the solar system. Galilei used his telescope to study the planets and found further evidence for the heliocentric idea. In Denmark, Tycho Brahe (1546-1601) made astronomical observations with data of unprecedented precision. He did so without the telescope as the first records of telescopes date back to around 1608 AD.left:Tycho Brahe (1546-1601) - right: Sophia Brahe (1559-1643). From Wikimedia Commons
(L, R), public domain.Brahe initially studied law, but developed a keen interest in astronomy. He was heavily influenced by the solar eclipse of August 21 in 1560. The eclipse had been predicted via the theory of celestial motion at that time. However, the prediction was off by a day. This led Brahe to the conclusion that in order to advance celestial science, many more and much better observations were needed. He devoted much of his time in achieving this. One of his best assistants was his younger sister, Sophie.On November 11 1572, Brahe observed a bright, new star in the constellation Cassiopeia (it consists of five bright stars forming a M or W). That was another event that made him decide to spend his days (or rather nights 😊) gathering astronomical data. The general believe in those days was still that everything beyond the Moon was eternal, never changing. So, this new star, that all in a sudden appeared, must be closer to the earth than the Moon itself. Brahe set out to measure its daily parallax against the five stars of Cassiopeia. But he didn't observe any parallax. Consequently, the new star's position had to be farther out than the Moon and the other planets that did show daily

> parallax. Moreover, Brahe kept measuring for months and still found no parallax. That meant that this new star was even further out than the known planets that show no daily parallax but did so for periods of month. Brahe reached the conclusion that this new 'thing' thus could not be yet another planet, but that it was a star. Another nail to the coffin of the Aristotle view. Brahe wrote a small book about it, called De Nova Stella (published in 1573). He uses the term 'nova' for a new star. We see this back in our name for the phenomenon observed by Brahe: we call it a supernova. By now it is known that this new star, this supernova is some 8,000 light years away from us. Brahe was upset by those who denied the new findings. In his introduction of De Nova Stella he writes (given here in our modern words): "Oh, coarse characters. Oh, blind spectators of heaven". The work and the booklet made his name in Europe as a leading scientist and astronomer.In the winter of 1577-1578 a comet, known as the "Great Comet" appeared in the skies. It was observed by many all over the globe (from the Aztecs in the America's via European researchers to the Arabic world, India all the way to Japan). Brahe made thousands of recordings, some simultaneously done in Denmark (close to Copenhagen) and Prague. That way, Brahe could establish that the comet was much beyond the Moon.At the end of his life, Brahe moved to Prague to become the official imperial astronomer under the protection of Rudolf II, the Holy Roman Emperor. In the later part of his life, Brahe had Johannes Kepler as his assistant.Kepler was 6 years old when the Great Comet appeared in the sky. He recorded in his writings that his mother had taken him to a high place to look at it. At the age of nine, he witnessed a lunar eclipse in which the Moon is in the Earth shadow, darkening it and turning quite red. As a child he suffered from smallpox making his vision weak and limited ability to use his hands. This made it difficult for him to make astronomical observations. It pushed him to mathematics. But there he was confronted with the Ptolemaic and the Copernican view on planetary motion. Kepler became a math professor at the Protestant Stiftsschüle in Graz. He wrote his ideas about the universe, following the thoughts of Copernicus in a book, that was read by Tycho Brahe. This brought him into contact with Brahe. In 1600 Kepler and his family moved to Prague as a consequence of political and religious oppression. He was appointed as assistant to Brahe and worked with Brahe on a new star catalogue and planetary tables. Brahe died unexpectedly on October 24th 1601. Two days later, Kepler was appointed as his successor.Johannes Kepler (1571-1630). From Wikimedia Commons, public domain.Kepler worked on a heliocentric version of the universe and in the period 1609-1619 published his first two laws. With these, he changed from trying circular orbits to other closed ones, to arrive at an elliptical one for Mars. That one was in very good agreement with the Brahe data, much better than had been achieved before. Kepler realized that the other planets might also be in elliptical orbits. In comparison with Copernicus he stated: the planetary orbits are not circles with epi-circles. Instead it are ellipses. Secondly, The sun is not at the center of the orbit, but in one of the focal points of the ellipse. Thirdly, the speed of a planet is not a constant.Kepler's work was not immediately recognized. On the contrary, Galilei completely ignored it and many criticized Kepler for introducing physics into astronomy.

Kepler has formulated three laws that describe features of the orbits of the planets around the sun.

1. The orbit of a planet is an ellipse with the Sun at one of the two focal points.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time (Law of Equal Areas).
3. The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.

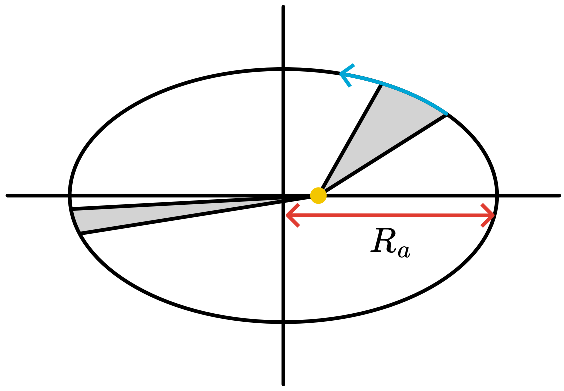


Figure 2.183: Kepler's 2nd Law of Equal Area.

$$\frac{T_A^2}{R_A^3} = \frac{T_B^2}{R_B^3} = const. \tag{2.253}$$

```
interactive(children=(IntSlider(value=1, description='t_sim', max=27374,
step=1825), Output()), _dom_classes=(…
```

```
<function __main__.sim_kep(t_sim)>
```

It is important to realize, that Kepler came to his laws by -what we would now call- curve fitting. That is, he was looking for a generic description of the orbits of planets that would match the Brahe data. He abandoned the Copernicus idea of circles with epi-circles with the sun in the center of the orbit. Instead he arrived at ellipses with the sun out of the center, in one of the focal points of the ellipse.

But, there was no scientific theory backing this up. It is purely 'data-fitting'. Nevertheless, it is a major step forward in the thinking about our universe and solar system. It radically changed from the idea that the universe is 'eternal', that is for ever the same and build up of circles and spheres: the mathematical objects with highest symmetry showing how perfect the creation of the universe is.

Kepler had formulated his laws by 1619 AD. It would take another 60 years before Isaac Newton showed that these laws are actually imbedded in his first principle approach: all that is needed is Newton's second law and his Gravitational Law.

2.4.6 Newton's theory and Kepler's Laws

The planets move under the influence of the gravitational force between them and the sun. We start with inspecting and classifying the force of gravity. Newton had formulated the Law of gravity: two objects of mass m_1 and m_2, respectively, exert a force on each other that is inversely proportional to the square of the distance between the two masses and is always attractive. In a mathematical equation, we can make this more precise:

$$\vec{F}_g = -G\frac{m_1 m_2}{r_{12}^2}\hat{r}_{12} \tag{2.254}$$

In the figure below, the situation is sketched. We have chosen the origin somewhere and denote te position of the sun and the planet by $\vec{r}_1$ and $\vec{r}_2$. Gravity works along the vector $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$. The corresponding unit vector is defined as $\hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}}$.

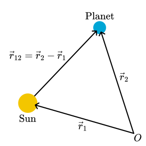


Figure 2.184: The sun and a planet.

Newton realized that he could make a very good approximation. Given that the mass of the sun is much bigger than that of a planet, the acceleration of the sun due to the gravitational force of the planet on the sun is much less than the acceleration of the planet due to the sun's gravity. For this, we only need Newton's 3rd law:

$$F_{g,sunonplanet} = -F_{g,planetonsun} \tag{2.255}$$

Hence

$$m_{sun}a_{sun} = -m_{planet}a_{planet} \to a_{sun} = \frac{m_{planet}}{m_{sun}}a_{planet} \ll a_{planet} \tag{2.256}$$

Newton concluded, that for all practical purposes, he could treat the sun as not moving. Next, he took the origin at the position of the sun. And from here on, we can ignore the sun and pretend that the planet feels a force given by

$$\vec{F}(\vec{r}) = -G\frac{mM}{r^2}\hat{r} \tag{2.257}$$

with M the mass of the sun and m that of the planet. r is now the distance from the planet to the origin and $\hat{r}$ the unit vector pointing from the origin to the planet.

First observation: The force is central!

First conclusion: Then the angular momentum of the planet is conserved (is a constant during the motion of the planet) and the motion is in a plane, i.e. we deal with a 2-dimensional problem!

Second Observation: The force is of the form $\vec{F}(\vec{r}) = f(r)\hat{r}$

Second conclusion: Thus, we do know that a potential energy can be associated with it. It is a conservative force. This also implies that the mechanical energy of the planet, that is the sum of kinetic en potential energy, is a constant over time. In other words, there is no frictional force and the motion can continue forever. This seems to be inline with our observation of the universe: the time scales are so large that friction must be small.

2.4.6.1 Constant Angular Momentum: Equal Area Law

The first clue towards the Kepler Laws comes from angular momentum. Let's consider the earth-sun system (ignoring all other planets in our solar system). As we saw above, gravity with the sun pinned in the origin, is a central force and thus

$$\frac{d\vec{l}}{dt} = \vec{r} \times \left(-G\frac{mM}{r^2}\frac{\vec{r}}{r}\right) = 0 \tag{2.258}$$

Thus, $\vec{l} = const.$ both in length and in direction. From the latter, we can infer that the motion of the earth around the sun is in a plane. Hence, we deal with a 2-dimensional problem.

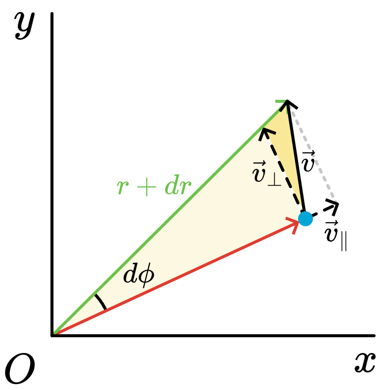


Figure 2.185: A free body diagram to help determine the area.

At some point in time, the earth is at location $\vec{r}$ (see red arrow in Figure 21). It has velocity $\vec{v}$, given by the black arrow. In a small time interval dt, the earth will move a little and arrive at $\vec{r} + d\vec{r}$ (the green arrow). As the time interval is very short, we can treat the velocity as a constant and thus write: $d\vec{r} = \vec{v}dt$.

Instead of concentrating on the motion of the earth, we focus on the area traced out by the earth orbit in the interval dt. That is the yellow area in Figure 21. This area is composed of two parts: the light yellow part and a smaller, bright yellow part. The light yellow part has an area $A_1 = \frac{1}{2} height x base$. If we make dt very small, the height is almost equal to r and the base becomes $v_\perp dt$ and thus $A_1 \approx \frac{1}{2} r v_\perp dt$. For the smaller yellow triangle we have: $A_2 = \frac{1}{2} dr x base \approx \frac{1}{2}\left(v_{//} dt\right) \cdot (v_\perp dt) = \frac{1}{2} v_{//} v_\perp dt^2$.

The total area traced out by the earth orbit during dt is thus in good approximation:

$$dA = A_1 + A_2 = \frac{1}{2}\left(r v_\perp + v_{//} v_\perp dt\right) dt \tag{2.259}$$

We divide both sides by dt and take the limit $dt \rightarrow 0$:

$$\frac{dA}{dt} = \left(\frac{1}{2} r v_\perp + \frac{1}{2} v_\perp v_/ / dt\right) \rightarrow \frac{1}{2} r v_\perp \tag{2.260}$$

In stead of $v_\perp$ we can also write $\frac{p_\perp}{m}$:

$$\frac{dA}{dt} = \frac{1}{2m} r p_\perp \tag{2.261}$$

But $r p_\perp$ is the magnitude of $\vec{r} \times \vec{p}$. And that is the magnitude of the angular momentum: $l = | | \vec{r} \times \vec{p} | | = r p_\perp$!!!

We know l is constant, thus we have found:

$$\frac{dA}{dt} = \frac{1}{2m} r p_\perp = \frac{l}{2m} = constant \tag{2.262}$$

We can easily integrate this equation:

$$\frac{dA}{dt} = \frac{l}{2m} \rightarrow A(t) = \frac{l}{2m} t + C \tag{2.263}$$

We can set the constant C to zero at some point in time t_0 and start counting the increase of the swept area. But we immediately infer that if we check the swept area between t and $t + \Delta t$, this will be $\Delta A = \frac{l}{2m} \Delta t$ regardless of where the earth is in its orbit. In words: in equal time intervals, the earth sweeps an area that is the same for any position of the earth. We have established the Equal Area Law!

2.4.6.2 Newton's theory and Kepler's Laws - part 2

We have:

- The sun is replaced by a force field originating at the origin. This force field is a central force.
    1. Thus, the angular momentum is conserved.
    2. The orbit is in a plane: we deal with a 2-dimensional problem.
- The force is conserved: a potential exists.

Based on these, we will derive Kepler's laws only using Newtonian Mechanics. This is easiest in polar coordinates (r, ϕ). However, in this course we do not deal with these coordinates. We will thus give a coarse overview of the steps that should be taken.

The first thing we notice, is that the constant angular momentum provides a constraint on the relation between $\vec{r}$ and $\vec{p}$. This constraint can be used to rewrite the kinetic energy $E_{kin} = \frac{1}{2} m v^2$ into $E_{kin} = \frac{1}{2} m \dot{r}^2 + \frac{l^2}{2mr^2}$.

What does this mean? The coordinate r is the distance from the sun to the earth. Its time derivative $\left(\dot{r} = \frac{dr}{dt} = v_r\right)$ is the velocity of the earth away from the sun. This is called the radial component of the velocity. Figure 22 illustrates this.

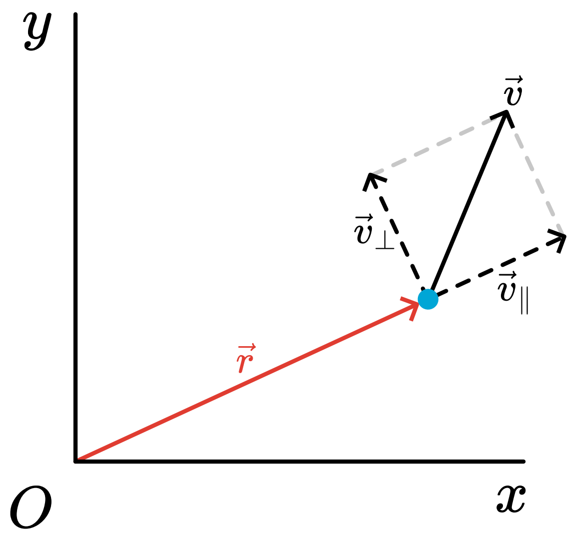


Figure 2.186: The coordinate r is the distance from the sun to the earth. Its time derivative $\left(\dot{r} = \frac{dr}{dt} = v_r\right)$ is the velocity of the earth away from the sun.

It is important to realize that $\dot{r}$ tells us if we are moving such that we are getting closer to the sun or further away. But it does not tells us how we move 'around' the sun. For that we need the information of the component of the velocity perpendicular to $\vec{r}$ (the other dashed vector in the figure).

So, it seems that we are working with incomplete information. And in a sense we do. But it will turn out to be very useful to understand the physics of the earth's orbit.

We already saw that in this case gravity is a conservative force. The potential energy is found by solving $V(r) = -\int_{r_{ref}}^{r} \vec{F}_g \cdot d\vec{r}$. We can plug in $\vec{F}_g = -G\frac{mM}{r^2}\hat{r}$. Thus only the radial coordinate is of importance in the inner product in the integral. Furthermore, we will use as reference boundary: ∞. Thus, the potential energy is:

$$\begin{aligned} V(r) &= -\int_{r_{ref}}^{r} \vec{F}_g \cdot d\vec{r} \\ &= GmM \int_{\infty}^{r} \frac{dr}{r^2} \\ &= -G\frac{mM}{r} \end{aligned} \tag{2.264}$$

Thus, energy conservation can be written as:

$$\frac{1}{2}m\left(v_x^2 + v_y^2\right) - -G\frac{mM}{r} = E_0 = const \tag{2.265}$$

As expected: we have an equation with two unknowns $(x(t), y(t))$. Once we solved the problem, we will thus have the coordinates of the planet's trajectory as a function of time. However, we will not do that. Reason: it is complicated and we don't need it! What we need is to find what kind of figure the trajectory is.

Our first step is to bring the number of unknowns in the energy equation down from two to one. For that, we use $E_{kin} = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2}$.

$$\frac{1}{2}\dot{r}^2 + \frac{l^2}{2mr^2} - -G\frac{mM}{r} = E_0 = const \tag{2.266}$$

Now we have an equation with only one unknown $r(t)$.

We can interpret this in a different way: the second term, with the angular momentum, originates from kinetic energy, but now looks like a potential energy. And that is exactly what we are going to do: treat it as a potential energy.

Hence, we can first inspect the global features of our energy equation. Notice that the gravity potential energy is an increasing function of the distance from the planet to the sun (located and fixed in the origin). This shows that the underlying force attractive is. The new part, coming from angular momentum, on the other hand is a decreasing function of distance. Thus, the related force is repelling.

We can make a drawing of the energy. See Figure 23.

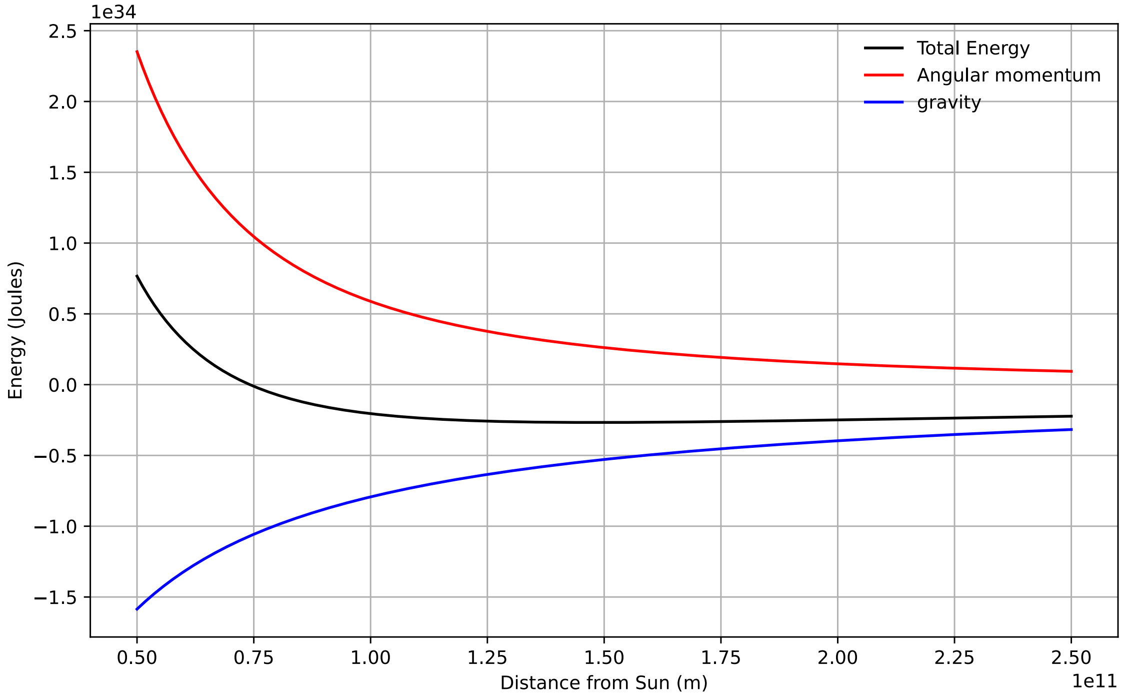


Figure 2.187: Energies related to our planet, with a minimum around $1.5e11m$.

The blue line is the potential energy of gravity. The red one stems from the kinetic energy associated with the angular velocity. The black line is the sum of the two, a kind of effective potential:

$$U_{eff} = \frac{l^2}{2mr^2} - -G\frac{mM}{r} \tag{2.267}$$

We see, that the energy cannot be just any value: the kinetic energy of our quasi-one-dimensional particle ($\frac{1}{2}m\dot{r}^2$) cannot be negative and the total potential energy has, according to Figure 23 a clear minimum. The total energy cannot be below this minimum. On the other hand: there is no maximum.

Case 1: Effective potential = minimal

Suppose, we would prepare the system such that its total energy was equal to the minimum of the black line, i.e. of the total potential energy. Then, of course, via the arguments we have given above this is only possible if the kinetic energy is zero.

$$E_{kin} + U_{eff}(r) = U_{eff,min} \Rightarrow E_{kin} = 0 \tag{2.268}$$

This implies that $\dot{r} = 0$:

$$E_{kin} = \frac{1}{2}m\dot{r}^2 = 0 \rightarrow \dot{r} = 0 \tag{2.269}$$

At first glance, this seems strange: $\dot{r} = 0$ suggests that the earth doesn't move, it has zero velocity. That would indeed be strange: after all we are dealing here with a planet that is attracted via gravity towards the sun. How can it possible have zero velocity?

We are about to make a mistake: $\dot{r} = 0$ doesn't mean that the velocity is zero. It means that $r(t) = const$. The planet still has a velocity perpendicular to its position vector $\vec{r}$. Earlier we found: $l = mrv_\perp = const$. We now have, since

$$\dot{r} = 0 \rightarrow r = r_0 = const, l = mr_0 v_\perp = const \rightarrow v_\perp = \frac{l}{m_{r_0}} = const \quad (2.270)$$

Thus, if a planet orbits its sun such that its (pseudo-)potential $U_{eff} = minimum$, then its orbit is a circle of radius r_0 that corresponds to the minimum in U_{eff} and the planet has a velocity that is constant in magnitude $v = \frac{l}{mr_0}$.

Case 2: Effective potential < Total energy < 0

Next, we consider a case where the total energy of the planet has a value between the minimum of the curve of the effective potential and 0. Call the value of the energy E_2.

From Figure 24 we see that the planet will now be confined to an area where the effective potential is either equal to or smaller than this particular value E_2

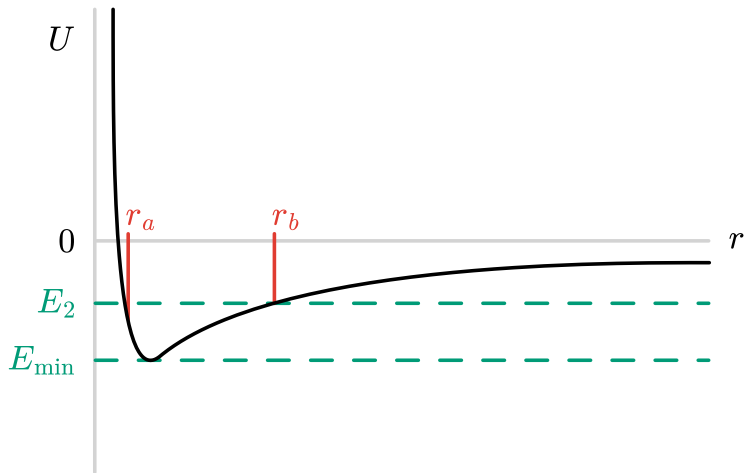


Figure 2.188: Total energy between 0 and minimum of effective potential.

Thus, the trajectory is confined between $r = r_a$ and $r = r_b$. At both these end points, the planet will have zero radial velocity: $\dot{r}_a = \dot{r}_b = 0$. However, as before, the planet will still have angular momentum and thus still have a non-zero velocity. The planet will travel in the (x, y)-plane between $r = r_a$ and $r = r_b$. How? We don't know yet.

N.B. Do realize, that the velocity is for this case not a constant. We already have established that it is linked to the angular momentum (which is a constant) and the distance to the origin.

Thus, if the planet is closer to r_a it moves faster than close to r_b. But it cannot escape from $r_a < r(t) < r_b$.

Case 3: Total energy > 0

Finally, we take the case that the total energy of the planet is positive, say a value of E_3 in Figure 25. Now we see that the planet can approach the sun, but not closer than a distance $r = r_c$. The planet is attracted to the sun, but after reaching the closest distance $r = r_c$ it will move away and eventually reach infinity. Again note: at $r = r_c$, the planet does have a non-zero velocity.

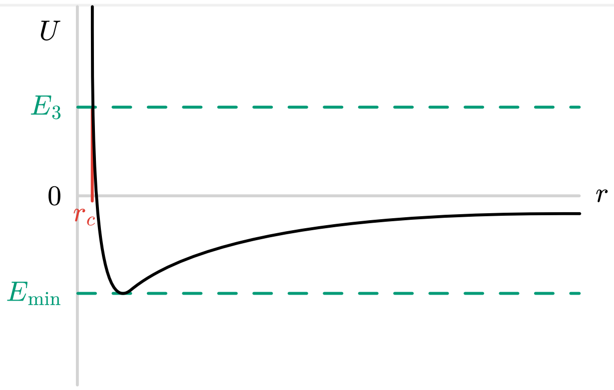


Figure 2.189: Total energy larger than 0.

2.4.6.3 Ellipsoidal orbits

We are left with the task of showing that planets 'circle' the sun in an ellipse. From the above, we now know that this must mean that the total energy is smaller than zero: $E < 0$. We will not go over the details of the derivation, but leave that for another course.

The outcome of the analysis would be the following expression for the orbit in case of an ellipse:

$$\frac{(x+ea)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2.271}$$

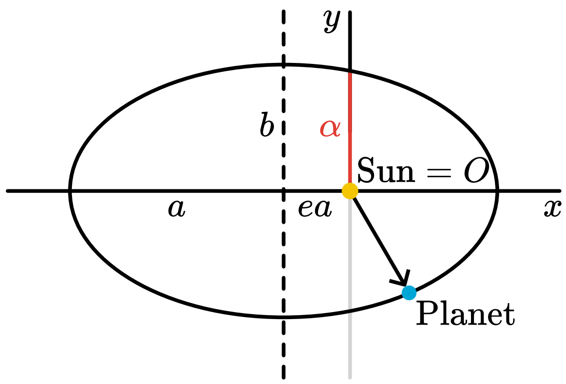


Figure 2.190: Ellips in Cartesian coordinates.

This is an ellipse with semi major and minor-axis a and b, respectively. The center of the ellipse is located at $(-ea, 0)$. Note that the sun is in the origin and that seen from the center of the ellipse, the origin is at one of the focal points of the ellipse. Consequently, the orbit is not symmetric as viewed from the sun. We notice this on earth: the summer and winter (when the sun is closest respectively furthest from the sun) are not symmetric, even if we take the tilted axis of the earth into account.

The half and short long axis are given by:

$$a = \frac{\alpha}{1-e^2} = \frac{GMm}{2\mid E\mid} \tag{2.272}$$

$$b^2 = a\alpha = \frac{l^2}{2m\mid E\mid} \tag{2.273}$$

with

$$e = \sqrt{1 + \frac{2El^2}{(GMm)^2 m}} \tag{2.274}$$

and

$$\alpha \equiv \frac{l^2}{2GMm^2} \tag{2.275}$$

This type of curve is know as the conic sections. That is, they can be found by intersecting a cone with a plane. See the animation below, where a plane is at various positions and at various angles intersecting a cone.

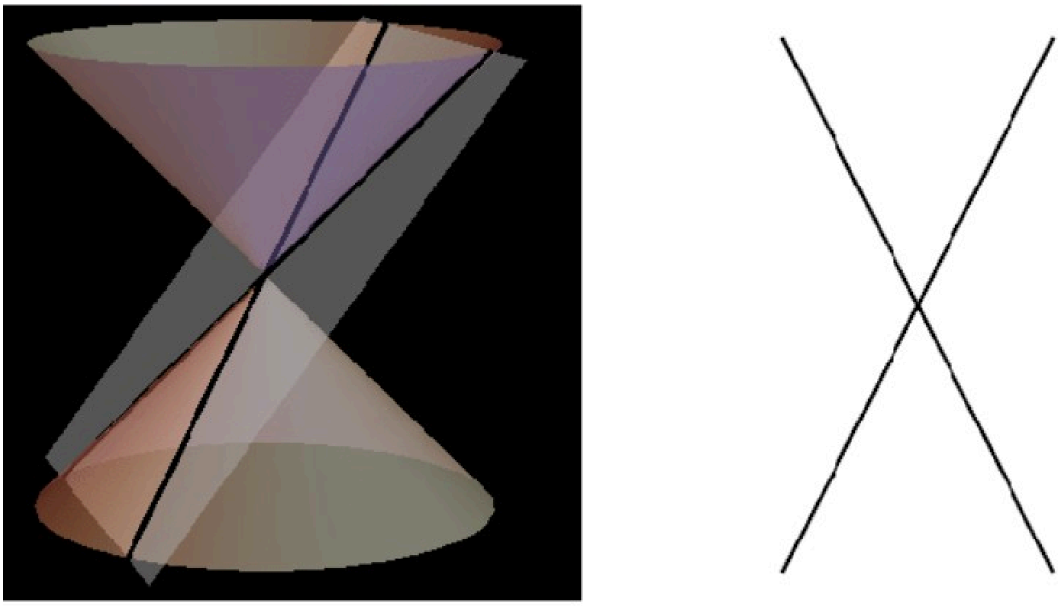


Figure 2.191: Conic sections animation created by Sara van der Werf, used with permission.

Note that in the definition of e, the total energy of the system plays a role. This energy can be negative (see Figure 23). The minimum value of the effective potential energy is easily computed. It is $U_{eff,min} = -\frac{1}{2}\frac{(GmM)^2 m}{l^2}$ and is realized when the planet is at a distance $r = \frac{l^2}{GMm^2}$. For this case we have $e = 0$ and the planet is moving in a circle around the sun, as we already argued above.

For $0 \leq e < 1$ the orbit is an ellipse as Kepler already had postulated (for these values of e the orbit is a closed one).

For $e = 1$, the orbit is a parabola: the object will eventually move to infinity where it has exactly zero radial velocity.

Finally, for $e > 1$ the trajectory is a hyperbola with the planet again moving to infinity.

Conclusion: according to Newton's laws of mechanics, combined with the Gravitation force proposed by Newton, planets must move in ellipses around their star.

This holds for our solar system, but for any other star with planets as well. Research has shown that there are hundreds of solar systems out in the universe with thousands of planets moving around their star. See e.g. https://exoplanets.nasa.gov/

2.4.6.4 Kepler 3

We are left with proving Kepler's third law:

$$\frac{T_A^2}{R_A^3} = \frac{T_B^2}{R_B^3} = const \tag{2.276}$$

Now that we know the orbit, this is not difficult. We concentrate on the motion during one lapse (one 'year'). From Keppler's 1^st law we know that the area a planet sweeps out of its ellipse is given by

$$A(t) = \frac{l}{2m}t + C \tag{2.277}$$

where C is an integration constant. Furthermore, this way of writing makes that the area swept keeps increasing: after one round along the ellipse, we simply keep counting.

However, we can easily back out what happens after exactly one round, or one 'year'. The total area swept is then, of course, the area of the ellipse itself, that is: in one year (time T) the area swept is πab. Hence we conclude:

$$A(T) = \pi ab \Rightarrow \pi ab = \frac{l}{2m}T \tag{2.278}$$

If we put back what we found for a and b, we get

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM} \tag{2.279}$$

Thus, indeed Kepler was right. Moreover, we note that the constant is only depending on the mass of the sun. The same law will hold for other solar systems, but with a different constant.

In Figure 28 Kepler's third law is shown for our solar system. The red data points are based on the measured 'year' of each planet and the distance to the sun. The blue line is the prediction from Newton's theory.

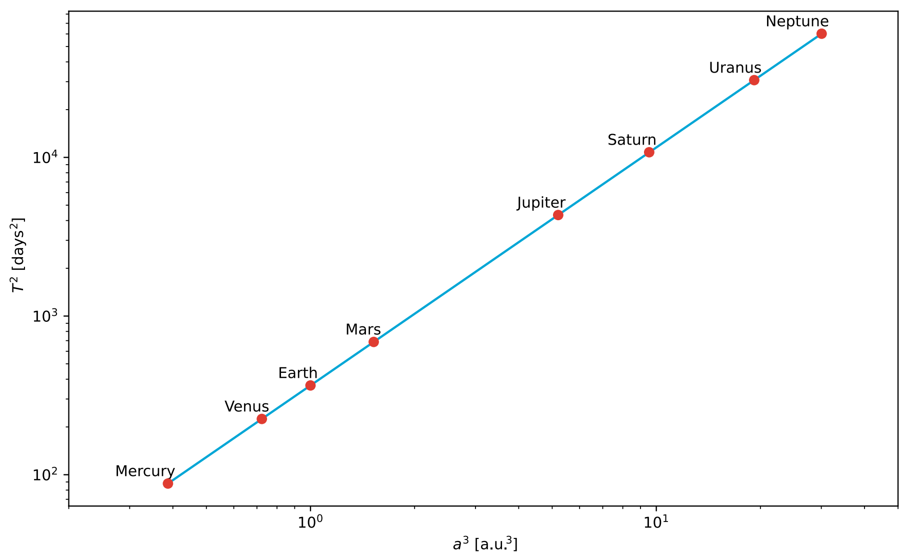


Figure 2.192: Kepler 3 for our solar system.

Haley's comet

The planets aren't the only objects that move around the sun. Several icy, rocky smaller objects are trapped in a closed orbit around the sun. These objects, comets from the Greek word for 'long-haired star', are left-overs from when our solar system was formed, some 4.6 billion years ago. There are many comets in our solar system. More than 4500 have been identified, but there are probably much more. Usually the orbit of a comet, if it's a closed one, has a high eccentricity (i.e. close to 1). Moreover, their orbital period may be very long.

One of the best visible comets is Haley's comet. However, its orbital period is about 75 years. It last appeared in the inner parts of the Solar System in 1986. So, you will have to wait until mid-2061 to see it again.

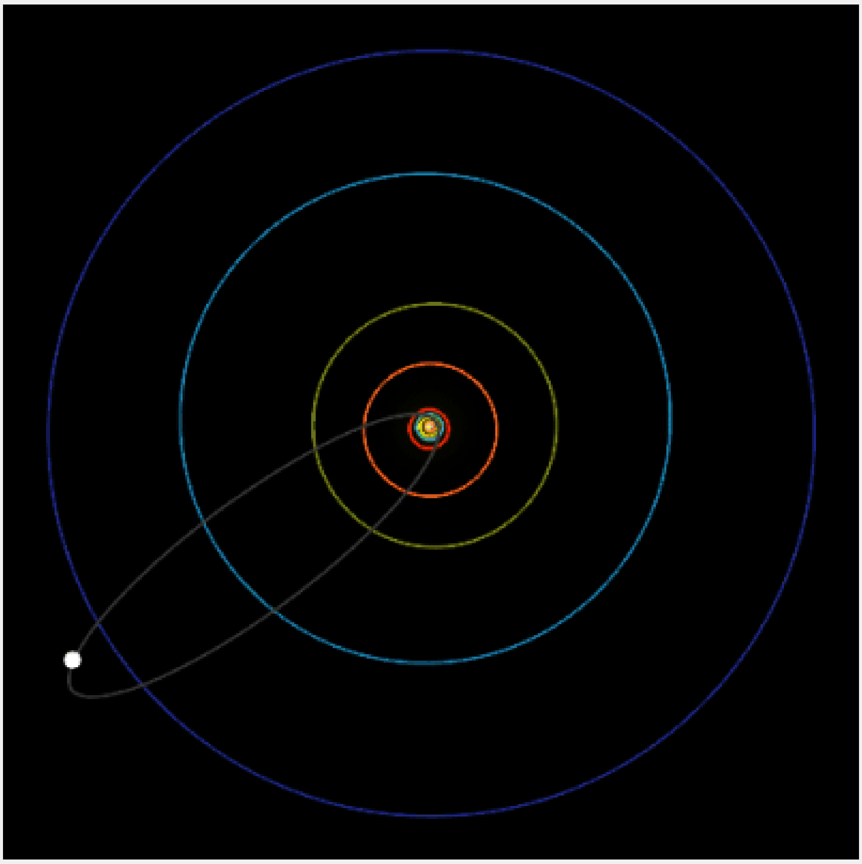


Figure 2.193: Trajectory of Haley's comet. From Wikimedia Commons, licensed under CC-BY 4.0.

2.4.7 Speed of the planets & dark matter

Starting from Kepler 3, we can compute the orbital speed of a planet around the sun

$$\begin{aligned} T^2 &= \frac{4\pi^2}{GM} a^3 \\ \omega^2 &= \frac{GM}{a^3}, \quad T = \frac{2\pi}{\omega}, \omega = \frac{v}{r}, a \approx r \\ \Rightarrow v &= \sqrt{\frac{GM}{r}} \end{aligned} \tag{2.280}$$

Indeed if we measure the speed of the planets in the solar system this prediction holds, the velocity drops with the distance from the sun as $\propto r^{-1/2}$ (see figure). As M we use the mass of the sun here.

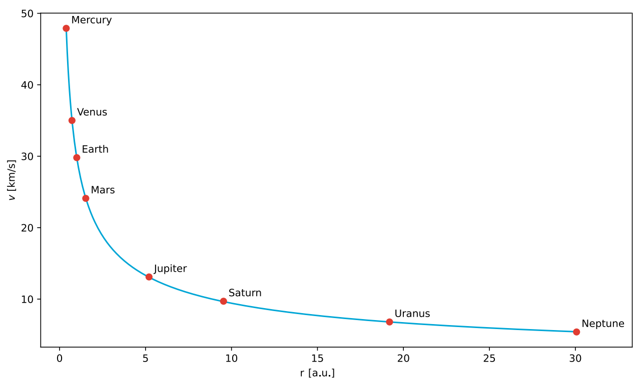


Figure 2.194: From LibreTexts Physics, licensed under CC BY-NC-SA 4.0.

The distance is measured in Astronomical Units [AU], the distance from the earth to the sun (about 8.3 light minutes). Note that the earth is moving with an unbelievable 30 km/s, that is 10^5 km/h! Do you notice any of that? We will use this motion later with the Michelson-Morley experiment.

If we plot the same speed versus distance curve not for the planets in our solar system, but for stars orbiting the center of our galaxy, the milky way, then the picture looks very different. The far away stars orbit at a much higher speed than expected and the form of the found curve does not match $\propto r^{-1/2}$.

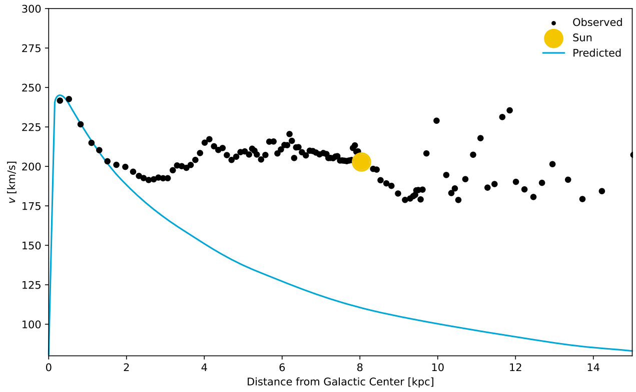


Figure 2.195: Adapted from Wikimedia Commons, licensed under CC-SA 3.0.

This mismatch is not understood to this day! The mass M here is calculated from the visible stars and the supermassive black holes at the center of the galaxy. But even if the mass is calculated wrongly, the shape of the dependency does not match. It turns out, this mismatch is observed in all galaxies! Apparently the law of gravity does not hold for large distances *or* there must be extra mass that increases the speed that we do not see. This mismatch has lead to the postulation of *dark matter* and an *alternative formulation* for the laws of gravity. This is the most disturbing problem in physics today; second is probably the interpretation of measurement in quantum mechanics (collapse of the wave function/Kopenhagen interpretation of Quantum Mechanics; multiverse theories).

The majority of all matter in the universe is believed to be *dark*. And we have no clue what it could be! Most scientist even think it must be non-baryonic, that is, other stuff than our well-known protons or neutrons. It remains most confusing.

The usual distance unit for distances in astronomy outside the solar system is not light years (ly), but parsec [pc], or kpc, or Mpc. One parsec is about 3.3 ly (or 10^{13} km). Note: stars visible to the eye are typically not more than a few hundred parsec away. The Milky Way is perfectly visible to the naked eye as a band/stripe of “milk” sprayed over the night sky. But you cannot see it anywhere close to Delft, there is much too much light from cities and greenhouses. Go to Scandinavia in the winter (“wintergatan”) or any place remote where there are few people. The reason you see a “band” in the night sky, is that the Milky Way is a spiral galaxy, sort of pancake shaped, and you see the band in the direction of the pancake.

2.4.8 Examples, exercises and solutions

Updated: 04 feb 2026

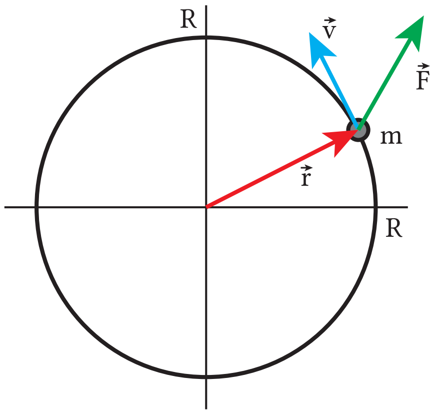
2.4.8.1 Worked examples

2.4.8.1.1 Circular motion

Consider a particle of mass m that is attached to a massless rope of length R. The other end of the rope is fixed in the origin and is free to rotate in a horizontal plane. The particle is made to move in a circular motion wit a velocity that has a constant magnitude v. Show that on m a force should act that is parallel to the rope at all times. Gravity is to be ignored.

Interpret the problem

Let's start with a drawing: m is moving over a circle in a horizontal plane. We draw its position vector, $\vec{r}$, its velocity vector, $\vec{v}$ and a force, $\vec{F}$ that can act on m.



This is clearly a 2-dimensional problem: both $\vec{r}$ and $\vec{v}$ do change in the plane while m moves along the circle. Moreover, since $\vec{v}$ is not a constant vector (it does have constant magnitude v, but clearly its direction constantly changes), we can anticipate that a force **must** be acting on m. After all, $\vec{p} = m\vec{v}$ is not a constant and, thus, according to N2 a force must act on m.

We have at least two options to approach this problem: via momentum or via angular momentum. We opt for the latter, as we anticipate that the angular momentum may be a constant.

Develop the solution

We are going to use angular momentum: $\vec{l} \equiv \vec{r} \times \vec{p}$. In this case $\vec{r}$ and $\vec{p} = m\vec{v}$ are always perpendicular (m moves over a circle and, hence $\vec{v}$ is tangent to the circle at all times). Thus we can write:

$$\vec{l} = \vec{r} \times m\vec{v} = mRv\hat{z} \tag{2.281}$$

with $\hat{z}$ a unit vector perpendicular to the plane of motion. Next we may use Newton 2 for angular momentum:

$$\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} \tag{2.282}$$

and from this law find restrictions on $\vec{F}$.

Evaluate the problem

Thus, we have for this case: $\vec{l} = mRv\hat{z}$: a constant vector as m, R, v are all constant. Consequently, from N2 for angular momentum we get:

$$\vec{l} = const \rightarrow \frac{d\vec{l}}{dt} = 0 \Rightarrow \vec{r} \times \vec{F} = 0 \tag{2.283}$$

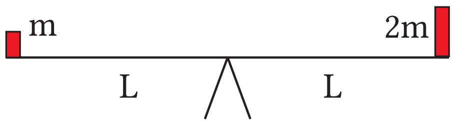
And we see that $\vec{F}$ must be parallel to $\vec{r}$.

Assess the problem

Our conclusion does make sense: the velocity of m is constant in magnitude, thus no force can act parallel to $\vec{v}$ since that would 'speed up' m. But a force is required to keep m in its circular orbit. After all, its velocity is changing with time and that requires a force acting on m.

2.4.8.1.2 Unstable See-Saw

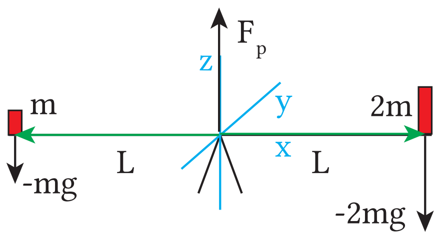
We have a seesaw as shown in the figure below. On the left side a mass m is place, on the right side, $2m$. Both arms of the seesaw have a length L and zero mass. For now, we keep the seesaw horizontal. But at $t = 0$, we let go.



1. Is the seesaw stable for $t > 0$?
2. If not: what is the initial acceleration of the mass $2m$ (that is, its acceleration just after release)?

Interpret the problem

As always, we start with a drawing. That is, in this case we complement the figure given with relevant information for our interpret-phase.



We have drawn: the two relevant forces of gravity acting om m and $2m$, respectively, as well as F_p the force of the pivot acting on the seesaw. Moreover, we have (in blue) indicated our coordinate system. This is useful, as we anticipate that we will have to deal with torques and angular momentum. Furthermore, we have drawn the position vector (in green) of both masses with the pivot point as our origin.

The figure is made with the idea that the stability of the seesaw for $t > 0$ can be inspected by looking at the torques acting on it.

Develop the solution

The seesaw is in balance if the sum of torques on it is zero. Thus, we calculate the net torque (using that the seesaw itself has no mass). We have chosen the pivot point as our origin, thus

$$\sum \vec{r}_i \times \vec{F}_i = -L\hat{x} \times -mg\hat{z} + 0 \times \vec{F}_p + L\hat{x} \times -2mg\hat{z} \tag{2.284}$$

If this is zero, then the seesaw will not start rotating.

If it is non-zero, the seesaw will start rotating and we analyze this by using $\frac{d\vec{l}}{dt} = \sum \vec{r}_i \times \vec{F}_i$

Evaluate the problem

Stable or not?

$$\sum \vec{r}_i \times \vec{F}_i = -L\hat{x} \times -mg\hat{z} + 0 \times \vec{F}_p + L\hat{x} \times -2mg\hat{z} = mgL\hat{y} \tag{2.285}$$

The net torques is clearly non-zero: the seesaw is unstable (for $t > 0$). It will start to rotate clockwise and we can find its initial acceleration as follows.

If we denote the velocity of m as $\vec{v}$ then $2m$ will have $-\vec{v}$ as velocity. This is of course a consequence of both sitting om the seesaw with equal distance to the pivot point. Furthermore, both velocities are always perpendicular to the position vector of their mass. Again this is a consequence of sitting on the seesaw, that can only rotate around the pivot point. Thus the angular momentum of the seesaw (with the two masses) is:

$$\vec{l} = \sum \vec{r}_i \times \vec{p}_i = -L\hat{x} \times mv\hat{z} + L\hat{x} \times -2mv\hat{z} = 3mvL\hat{y} \tag{2.286}$$

This is not a constant, as v is a function of time. From N2 for angular momentum

$$\frac{d\vec{l}}{dt} = \sum \vec{r}_i \times \vec{F}_i \tag{2.287}$$

we learn that for t small (i.e. gravity is still at 90° with the position vectors of m and $2m$)

$$3mL\frac{dv}{dt} = mgL \Rightarrow a \equiv \frac{dv}{dt} = \frac{1}{3}g \tag{2.288}$$

Thus, the initial acceleration is a third of g.

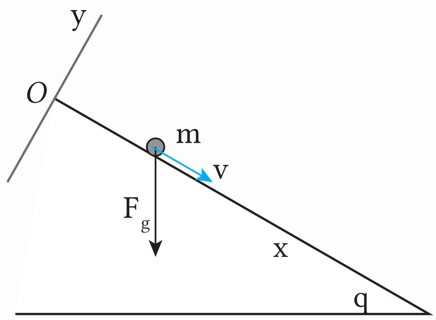
Assess the problem

Obviously, this seesaw is unstable. We knew that from experience. Our intuition is now backed up by a formal physics reasoning. Furthermore, we see that the initial acceleration is positive. Hence, the lighter mass m starts moving upwards, also in line with our intuition.

Finally, we have computed the initial acceleration and, indeed, our answer has the right units: m/s^2.

Exercise 2.199: Sliding down a slope: angular momentum

Consider a point particle of mass m sliding down a slope as shown in the figure below.



Show that the angular momentum of this particle, using the coordinate system given, is always zero. Do realize, that point particles have no size (in the drawing m seems to have a finite size, but that is for clarity in the drawing).

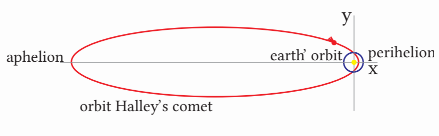
Provide arguments why this is not in conflict with gravity exerting a torque on m.

Exercise 2.201: Halley's comet

Halley's comet orbits the sun in a very elongated ellipse: $e = 0.967$. With the origin of a Cartesian coordinate system in the sun, the comet's orbit is given by:

$$\sqrt{x^2 + y^2} + ex = a(1 - e^2) \tag{2.289}$$

with $e = 0.967$ the eccentricity and $a = 17.8AU$ the half long axis of the ellipse. (Note: AU stands for 'astronomical unit' which is the averaged distance from the earth to the sun, i.e. about 149.6 million km.)



Since gravity is a central force (in this case we have fixed the sun in the origin), the angular momentum of Halley's comet is constant. Find the ratio of the velocity of the comet when passing the sun at its closest distance (the so-called perihelion) and the velocity at the furthest distance (the aphelion). Compare this ratio to that of the earth.

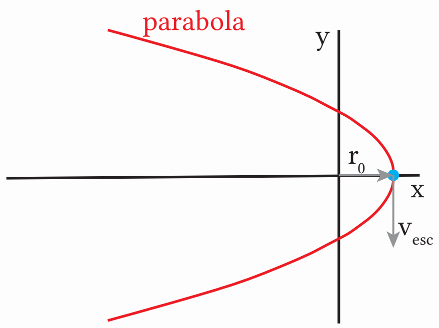
Data: the ellipse for an object orbiting the sun is given by

$$(x^2 + y^2) + ex = a(1 - e^2) \tag{2.290}$$

object	a (A.U.)	e (-)
earth	1.0	0.0167
Halley's comet	17.8	0.967

Exercise 2.203: Particle following a Parabola

An object that is in a central force field of the form $\vec{F} = -\frac{k}{r^2}\hat{r} can under very specific conditions follow a parabolic trajectory: its energy must be exactly zero.



a) Find the relation between the closest distance (r_0) to the force center (e.g. the sun in case of gravity on comets and planets) end the highest velocity, v_{esc} of the object.

b) Using (a), express the angular momentum of the object in terms of the closest distance to the force center.

Exercise 2.205: Relative distance to the sun

Measuring distances in astronomy is a difficult task: one can not use a standard measuring rule. However, there are plenty of other options. One is: use Kepler's third law to find the relative distance from the planets to the sun. Here 'relative' means: express them in terms of the distance of the earth to the sun.

Data

planet	orbital period (year)
mercury	0.241
venus	0.615
earth	1.000
mars	1.881
jupiter	11.86
saturn	29.46
Uranus	84.0
Neptune	164.8

Exercise 2.206: Giant Pirate Ship

In theme parks giant pirate ships suspended like a swing offer a thrilling experience. In terms of physics, we can think of these ships as very large swings. Take for example a swing of length $L = 15m$ (this mimics the pirate ship The Halve Maen in the theme park The Efteling in The Netherlands).

The swing has a maximum angle with the vertical of 90° and is released from this angle with zero initial velocity.

a) Find the maximum velocity of the swing.

b) Calculate the minimum and maximum angular momentum of a person (mass m) on the swing.

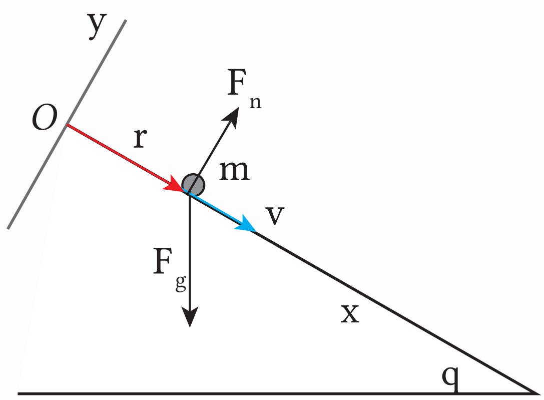
c) Which force(s) are contributing to the change in angular momentum?

2.4.8.2 Exercises

2.4.8.3 Answers

Solution 2.207: Solution to Exercise 1

We first complete the drawing. As we need to think about the angular momentum, we need the position vector of m. Furthermore, we need to include all forces acting on m in order to evaluate the effect of all torques on m.



Apart from gravity also a normal force from the slope on m is present.

Note: in the figure we have now for the position vector, the velocity and both forces set the size of m to zero. As a consequence, $\vec{r}$ and $\vec{v}$ are parallel during the entire motion

Next step, we develop our solution strategy. As this exercise is concerned with angular moment, we will use N2 for angular momentum as well as, of course, the definition of angular momentum $\vec{l} \equiv \vec{r} \times \vec{p}$.

Now we evaluate our ideas:

$$\vec{l} \equiv \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} = 0 \tag{2.291}$$

since, in this case $\vec{r}//\vec{v}$.

N2 for angular momentum reads as:

$$\frac{d\vec{l}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i \tag{2.292}$$

We know that the left hand side is zero: $\vec{l}$ is constant. Thus, the sum of all torques on m must also be zero. We have identified two forces acting on m: gravity and the normal force. The latter does nothing but opposing the component of gravity perpendicular tot he slope (that is its y-component). We split gravity in its x and y-component:

$$\vec{F}_g = F_{g,x}\hat{x} + F_{g,y}\hat{y} \tag{2.293}$$

with this we compute the total torque on m

$$\begin{aligned}\sum_i \vec{r}_i \times \vec{F}_i &= \vec{r} \times \left(F_{g,x}\hat{x} + F_{g,y}\hat{y}\right) + \vec{r} \times \vec{F}_n \\ &= \underbrace{\vec{r} \times F_{g,x}\hat{x}}_{=0} + \vec{r} \times \underbrace{\left(F_{g,y}\hat{y} + \vec{F}_n\right)}_{=0} \\ &= 0\end{aligned} \tag{2.294}$$

The first term is zero because $\vec{r}//\hat{x}$ and the second one due to cancelling of the two forces in the brackets.

If we assess our answer, we see that this is perfectly in line with N2 for angular momentum: no net torque, hence no change of the angular momentum!

Solution 2.209: Solution to Exercise 2

We will exploit that the angular momentum of the earth and the comet are constant. The angular momentum is defined as $\vec{l} = \vec{r} \times \vec{p}$. At the perihelion and at the aphelion, the position vector and the velocity are perpendicular. Thus, the angular momentum simplifies to $\vec{l} = x_h m v \hat{z}$, with x_h the x-coordinate of the perihelion or aphelion.

We can find these coordinates by realizing that at the perihelion and the aphelion, the y-coordinate is zero. Thus,

$$\begin{gathered} \sqrt{x_h^2 + 0^2} + e x_h = a(1 - e^2) \rightarrow \\ \pm x_h + e x_h = a(1 - e^2) \rightarrow \\ x_{ph} = a(1 - e) or x_{ah} = -a(1 + e) \end{gathered} \tag{2.295}$$

where we have labelled the x-coordinate of the perihelion as x_{ph} and of the aphelion as x_{ah}.

The velocity ratio of v_{ph} over v_{ah} is now easily found, by using that the angular momentum is a constant:

$$l = m x_h v \Rightarrow \frac{v_{ph}}{v_{ah}} = \frac{x_{ap}}{x_{ph}} \tag{2.296}$$

Putting in the given numbers, we find:

$$\begin{aligned} \left[\frac{v_{ph}}{v_{ah}}\right]_{Halley} &= \frac{1 + e_H}{1 - e_H} = 59.6 \\ \left[\frac{v_{ph}}{v_{ah}}\right]_{e} &= \frac{1 + e_e}{1 - e_e} = 1.03 \end{aligned} \tag{2.297}$$

Does this make sense? Yes: the earth orbits in an ellipse that is close to a circle. Hence we expect only a mild difference between the velocity at the perihelion and the aphelion. For Halley's comet this is rather different. According to Kepler's law of equal area, the comets sweeps through the same area close to the sun (i.e. around the perihelion) and far away from the sun (the aphelion). As the distance to the sun close to the perihelion is much smaller than that around the aphelion, the velocity much be much smaller around the aphelion to ensure equal areas in equal times. Note: our expressions are dimensionally correct.

Solution 2.210: Solution to Exercise 3

The energy of the object is given by:

$$\frac{1}{2}mv^2 + V(r) = E_0 \tag{2.298}$$

We can find the potential from the force:

$$V(r) \equiv -\int_{\vec{r}_{ref}}^{\vec{r}} \vec{F} \cdot d\vec{r} = k\int_{\infty}^{r} \frac{dr}{r^2} = -\frac{k}{r} \tag{2.299}$$

where we have used that the force is central. Furthermore, we have taken as our reference point: infinity.

If the object follows a parabola, then its energy must be exactly zero. Thus:

$$\frac{1}{2}mv^2 - \frac{k}{r} = 0 \tag{2.300}$$

a) This holds at any point of the trajectory. Thus for the point of closest distance to the force center we get:

$$\frac{1}{2}mv_{esc}^2 - \frac{k}{r_0} = 0 \Rightarrow v_{esc} = \sqrt{\frac{2k}{mr_0}} \tag{2.301}$$

b) Now that we have both the velocity and the position at one point on the parabola, we can compute the angular momentum. Since the force is central, we do know that the angular momentum is a constant. Hence, computing it for one particular point will give us the angular momentum for all points on the trajectory.

We use that at the point of closest approach the position vector and the velocity are perpendicular: $\vec{r}_0 \perp \vec{v}_{esc}$. Thus the angular momentum is:

$$\vec{l} \equiv \vec{r} \times m\vec{v} = -mr_0 v_{esc}\hat{z} = -\sqrt{2kmr_0}\hat{z} \tag{2.302}$$

Note: the minus sign is needed to have the angular momentum point into the drawing as $\hat{z}$ points towards us.

Solution 2.211: Solution to Exercise 4

According to Kepler 3, the orbital period, T, and the length of the semi major axis, a are related as:

$$T^2 = ka^3 \tag{2.303}$$

with k a know constant, the same for all planets orbiting the sun. From this law, we immediately get that

$$\frac{a_{planet}}{a_{earth}} = \left(\frac{T_{planet}}{T_{earth}}\right)^{2/3} \tag{2.304}$$

Thus, if we want to find the relative distance to the sun of each planet (that is, we express a in terms of a_{earth} which is 1 A.U.), we get:

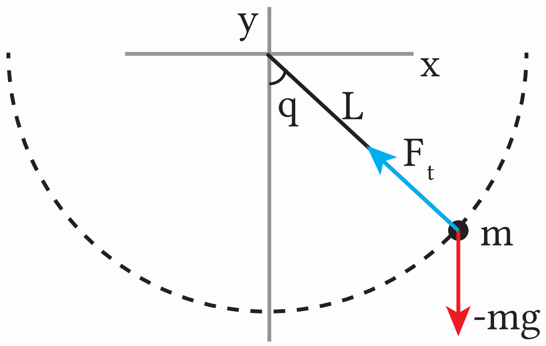
Data

planet	orbital period (year)	semi-major axis (A.U.)
mercury	0.241	0.39
venus	0.615	0.72
earth	1.000	1.00
mars	1.881	1.52
jupiter	11.86	5.20
saturn	29.46	9.58
Uranus	84.0	19.2
Neptune	164.8	30.1

Fun fact: Neptune's orbital period is so large, that since its discovery in 1846, it has only completed 1 Neptune-year (first full orbit in 2011!).

Solution 2.212: Solution to Exercise 5

First we make a sketch.



We have drawn gravity and the tension in the rod connecting the mass m tot the pivot point. From a physics point of view, the situation is as follows: the swing initially has only potential energy and it will gain kinetic energy on its way down, until all potential energy is converted into kinetic energy.

Two forces are acting on the swing, giving rise to two torques that cause the angular momentum to change. The velocity of the mass is always perpendicular to its position vector (a consequence of using a 'clever' choice for the origin of our coordinate system).

Now, we can set up the modeling for this problem.

Conservation of energy:

$$\frac{1}{2}mv^2 + mgy = const \tag{2.305}$$

Angular momentum:

$$\vec{l} = \vec{r} \times \vec{p} \Rightarrow l = mLv \tag{2.306}$$

Change of angular momentum:

$$\frac{d\vec{l}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i \tag{2.307}$$

With this, we are ready for the develop phase. Conservation of energy gives us for the velocity at the lowest point:

$$\frac{1}{2}mv_{max}^2 + 0 = mgH \Rightarrow v_{max} = \sqrt{2gL} = 22m/s \tag{2.308}$$

The minimum angular momentum is of course $l_{min} = 0$ at the highest point where the swing has zero velocity and changes direction.

Tha maximum angular momentum is at the lowest point: $l_{max} = mL\sqrt{2gL}$

Finally, the torque comes from gravity: $\vec{\Gamma} = \vec{r} \times -mg\hat{y} = mgL\sin\theta\hat{z}$. The tension in the rod is always parallel to the position vector of m and thus has zero torque.

2.5 Conservation Laws / Galilean Transformation

Updated: 04 feb 2026 In the previous chapters, we have seen that from Newton's three laws, we can obtain conservation laws. That means, under certain conditions (depending on the law), a specific quantity cannot change.

These conservation equations are very important in physics. They tell us that no matter what happens, certain quantities will be present in the same amount: they are *conserved.*

Conservation of energy follows from the concept of work and potential energy.
Conservation of momentum is a direct consequence of N2 and N3, as we will see below. And finally, under certain conditions, angular momentum is also conserved. In this chapter we will summarize them. The reason is: their importance in physics. These laws are very general and in dealing with physics questions they give guidance and very strict rules that have to be obeyed. They form the foundation of physics that cannot be violated. They provide strong guidance and point at possible directions to look for when analyzing problems in physics.

2.5.1 Conservation of Momentum

Consider two particles that mutually interact, that is they exert a force on each other. For each particle we can write down N2:

$$\begin{matrix} \frac{d\vec{p}_1}{dt} = \vec{F}_{21} \\ \frac{d\vec{p}_2}{dt} = \vec{F}_{12} = -\vec{F}_{21} \end{matrix} \} \to \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0 \Rightarrow \vec{p}_1 + \vec{p}_2 = const \tag{2.309}$$

The total (linear) momentum is conserved if only internal forces are present; "action-reaction pairs" always cancel out.
This law has no exception: it must be obeyed at all times. The total momentum is constant, momentum lost by one must be gained by others.

2.5.2 Conservation of Energy

As we have seen when deriving the concept of potential energy, for a system with conservative forces the total amount of kinetic and potential energy of the system is constant. We can formulate that in a short way as:

$$\sum E_{kin} + \sum V = const \tag{2.310}$$

Again: energy can be redistributed but it cannot disappear nor be formed out of nothing.

If non-conservative forces are present, the right hand side of the equation should be replaced by the work done by these forces.

$$\sum E_{kin} + \sum V = \sum W \tag{2.311}$$

In many cases this will lead to heat, a central quantity in thermodynamics and another form of energy. The "loss" of energy goes always to heat. With this 'generalization' we have a second law that must always hold. Energy cannot be created nor destroyed. All it can do is change its appearance or move from one object to another.

2.5.3 Conservation of Angular Momentum

Also angular momentum can be conserved. According to its governing law $\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F}$ it might seem that we can for two interacting particles again invoke N3 "action = -reaction" and the terms with the forces will cancel out. But we need to be a bit more careful, as cross products are involved which are bilinear (a type of mathematical function or operation that is linear in each of two arguments separately, but not necessarily linear when both are varied together). So, let's look at the derivation of "conservation of angular momentum" for two interacting particles:

$$\left.\begin{array}{r}\frac{d\vec{l}_1}{dt}=\vec{r}_1\times\vec{F}_{21}\\ \frac{d\vec{l}_2}{dt}=\vec{r}_2\times\vec{F}_{12}=-\vec{r}_2\times\vec{F}_{21}\end{array}\right\}\rightarrow\frac{d}{dt}\left(\vec{l}_1+\vec{l}_2\right)=(\vec{r}_1-\vec{r}_2)\times\vec{F}_{21} \tag{2.312}$$

As we see, this is only zero if the vector $\vec{r}_1-\vec{r}_2$ is parallel to the interaction forces (or zero). We called this a *central force*. Luckily, in many cases the interaction force works over the line connecting the two particles (e.g. gravity). In those cases, the angular momentum is conserved. Mathematically we can write this as:

$$if\vec{F}_{21}\,||\,(\vec{r}_1-\vec{r}_2) \quad\Rightarrow\quad \vec{l}_1+\vec{l}_2=const \tag{2.313}$$

Conservation of Mass

Within the world of Classical Mechanics, mass is also a conserved quantity. Whatever you do, what ever the process the total mass in the system stays the same. We cannot create nor destroy mass. From more modern physics we know that this is not true. On the one hand we can destroy mass. For instance, when an electron and a positron collide, they can annihilate each other resulting in two photons, i.e. 'light particles' that do not have mass. Similarly, we can create mass out of light. This is the inverse of the annihilation: pair production. If we have a photon of at least), then -under the right conditions- an electron-positron pair can be created.Moreover, Albert Einstein showed that mass and energy are equivalent - expressed via his famous equation . His theory of Relativity showed us that in collisions at extreme velocities mass is not conserved: it can both be created or disappear. Rephrased: it is actually part of the energy conservation, mass is in that context just a form of energy.

Emmy Noether, symmetries and conservation laws

We discussed the conservation laws as consequences of Newton's Laws. That in itself is ok. However, there is a deeper understanding of nature that leads to these conservation laws. And from the conservation laws we can go to Newton's Laws, thus 'reversing the derivations' and starting from this new, different way of looking at nature.What is it and how do we know? To answer this question we have to resort to Emmy Noether, a German mathematician. Noether made top contributions to abstract algebra. She proved, what we now call, Noether's first and second theorems, which are fundamental in mathematical physics. Noether is often named as one of the best if not the best female mathematicians ever lived. Her work on differential invariants in the calculus of variations has been called "one of the most important mathematical theorems ever proved in guiding the development of modern physics".Amalie Emmy Noether (1882-1935). From Wikimedia Commons, public domain.Noether shows, that if a dynamic system is invariant under a certain transformation, that is it has a symmetry, then there is a corresponding quantity that is conserved. Ok, pretty abstract. What does it mean, any examples? Yes! Here is one.In physics we believe that it does not matter if we do an experiment now and repeated it exactly under the same conditions an hour later, the outcome will be the same. Or rephrased: if we translate it in time, the outcome is the same; the laws of physics are invariant. This is in mathematical terms a symmetry, a symmetry with respect to time. Noether's theorem then shows, that there is a conserved quantity and this quantity is energy. Hence, based on the idea that time itself has no effect on physical laws, we immediately arrive at conservation of energy.Second example: we also believe that place or position in the universe doesn't matter. The physical laws are not only always the same (time invariance), they are also the same everywhere (space invariance). From this symmetry, via Noether's work, we immediately get that momentum is a conserved quantity. Thus, these two invariances or symmetries -time and space - provide us directly with conservation of energy and momentum and from that we could easily derive Newton's second and third law. Much of modern physics is now built on the ideas put forward by Emmy Noether. That goes from quantum mechanics to quarks to string theory.

2.5.4 Galilean Transformation

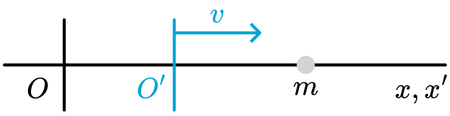
There is one important element of Classical Mechanics that we have to add: for which type of observer do Newton's Laws hold? The original thought was: for inertial observers. These are observers that are at rest with respect to an inertial frame of reference.

But this merely shifts the question to: what is an inertial frame of reference? One possible answer is: an inertial frame of reference is a frame in which Newton's Laws hold. That is: a particle on which, according to an observer in such a frame, no net force is acting will keep moving at a constant velocity.

All inertial frames of reference move at a constant velocity with respect to each other. They cannot accelerate. To picture what it means, an inertial frame of reference or an inertial observer, we sometimes use the idea that such a frame or observer moves at a constant velocity with respect to the 'fixed' stars. And indeed, for a long time people believed that the stars were fixed in space. But from more modern times we do know, that this is not the case: stars are not fixed in space nor do they move at a constant velocity.

Later in the study of Classical Mechanics, we will see, that it is possible to do without the restriction that Newton's Law strictly speaking only hold in inertial frames. But for now, we will stick to inertial frames and look at the 'communication' between two observers in two different inertial frames.

An important requirement of any physical law is that it looks the same for all inertial observers. That doesn't mean that the outcome of using such a law is the same. As a trivial example, take two inertial observers S and S'. According to S, S' moves at a constant velocity, V, in the x-direction. S' observes a mass m that is not moving in the frame of reference of S'. For simplicity, we will assume that each observer is in its own origin.



S' rightfully concludes, based on Newton's 1^{st} law that no force is acting on m. S agrees, but doesn't conclude that m is at rest. This is trival: both observers can use Netwon's second law which for this case states that $\frac{d\vec{p}}{dt} = 0 \rightarrow \vec{p} = const \rightarrow \vec{v} = const$. But the constant is not the same in both frames.

To make the above loose statements more precise. We have two coordinates systems CS and CS'. The transformation between both is given by a translation of the origin of S' with respect to that of S.

2.5.4.1 Communication Protocol

We need to have a recipe, a protocol that translates information from S' to S and vice versa.

This protocol is called the *Galilean Transformation* between two inertial frames, S and S'.

According to observer S, S' is moving at a constant velocity V. Both observers have chosen their coordinate system such that x and x' are parallel. Moreover, at $t = t' = 0$, the origins O and O' coincide. The picture below illustrates this.

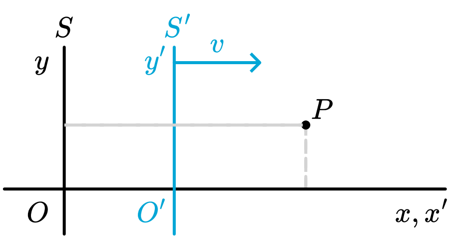


Figure 2.215: Two inertial observers S and S' and their coordinate systems.

Consider for simplicity a 2D point P with coordinates (x', y') and time t' for S'. What are the coordinates according to S? First of all: in classical mechanics, there is only one time, that is: $t = t'$. Until the days of Einstein this seemed self evident; we now know that nature is more complex.

For the spatial coordinates, we see immediately: $y = y'$. And for the x-coordinate S can do the following. To go to the x-coordinate of P, first S goes to the origin O' of S'. O' is a distance Vt from O. Thus, the distance to P along the x-axis is $Vt + x'$. If we sum the above up, we can formulate the relation between the coordinate system of the two observers. This transformation is the **Galilean Transformation**, or **GT** for short.

Galilean Transformation

$$\begin{aligned} x' &= x - Vt \\ y' &= y \\ t' &= t \end{aligned} \tag{2.314}$$

2.5.4.2 Velocity is relative; acceleration is absolute

A direct consequence of the Galilean Transformation is that velocity is observer-dependent (not surprising), but for observers in inertial frames, observed velocities differ by a constant velocity vector.

In what follows we will derive the relations between velocity and acceleration as observed by S and S'. Note that we need to be precise in our notation: S' denotes quantities with a prime ('), but S does not. This is obvious for the coordinates as S uses x whereas S' will write x'. It is, however, also wise to use primes on the velocity: S will denote the x-component as: $v_x = \frac{dx}{dt}$. So, S' will note denote velocity by v, but by v'. Hence S' will write $v'_{x'} = \frac{dx'}{dt'}$. Now, obviously, $t' = t$ so we could drop the prime on time, but it is handy to do that in the second step.

First we look at velocity.

$$\begin{aligned} v'_{x'} &\equiv \frac{dx'}{dt'} \Rightarrow v'_{x'} = \frac{d(x - Vt)}{dt} = v_x - V \\ v'_{y'} &\equiv \frac{dy'}{dt'} \Rightarrow v'_{y'} = \frac{dy}{dt} = v_y \end{aligned} \tag{2.315}$$

Thus indeed velocity is 'relative': different observers find different values, but they do have a simple protocol to convert information from the other colleague to their own frame of reference.

Secondly, we look at acceleration.

$$\begin{aligned} a'_{x'} &\equiv \frac{dv'_{x'}}{dt'} \Rightarrow a'_{x'} = \frac{d(v_x - V)}{dt} = a_x \\ a'_{y'} &\equiv \frac{dv'_{y'}}{dt'} \Rightarrow a'_{y'} = \frac{dv_y}{dt} = a_y \end{aligned} \tag{2.316}$$

So, we conclude: acceleration is the same for both observers.

Consequently, N2 holds in both inertial systems if we postulate that $m' = m$. In other words: mass is an object property that does not depend on the observer.

Thus, two observers, each with its own inertial frame of reference, will both *see the same forces*: $F = ma = m'a' = F'$.

This finding is stated as: Newton's second law is *invariant* under Galilean Transformation. Invariant means that the form of the equation does not change if you apply the Galilean coordinate transformation. Later we will expand this to Lorentz invariant transformation in the context of special relativity. The concepts of invariance is very important in physics as hereby we can formulate laws that are the same for everybody (loosely speaking).

2.5.5 Exercises, examples & solutions

Updated: 04 feb 2026

2.5.5.1 Worked examples

2.5.5.1.1 Title of example

Interpret the problem

HIER DE INTERPRETATIE

Develop the solution

HIER DE DEVELOPMENT

Evaluate the problem

HIER DE EVALUATE

Assess the problem

HIER DE ASSESS

2.5.5.2 Superballs

In class you have seen the *Superballs* example. Let's dive more deep into what is happening.

Figure 2.216: Watch the superballs again.

Consider Figure 2, if you let a smaller and a larger ball drop together, stacked on top of each other, the smaller ball will bounce back much stronger (higher) than if you let the small ball fall without stacking it on the lager ball. How can that happen?

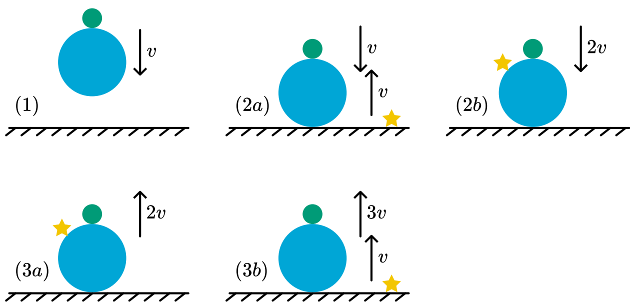


Figure 2.217: Bouncing balls.

To explain this we use the Galilean Transformation (GT). Consider the following situation depicted in Figure 2.

- **1** Both balls are falling with velocity $\vec{v}$ towards the ground.
- **2a** The larger ball just hit the ground. As the mass of the ground is much larger than that of the large ball, it is (elastically) reflected, i.e. the direction of the velocity is reversed but the magnitude stays the same. The small ball is still moving downwards with $\vec{v}$.
- **2b** We apply a GT of the observer (yellow star) from the ground to an observer moving with the larger ball. The observer moving with the larger ball sees the smaller ball moving with $2\vec{v}$ towards it.
- **3a** The smaller ball hits the larger ball and is reflected due to its smaller mass. In the frame of the observer on the larger ball, the smaller ball now moves with $2\vec{v}$ away from it.

- **3b** We apply a GT of the observer (yellow star) from the larger ball back to an observer on the ground. For the observer on the ground the larger ball has velocity $\vec{v}$ upwards from **2a**, therefore the smaller ball has velocity $3\vec{v}$ upwards.

The smaller ball has now velocity $3\vec{v}$ instead of $\vec{v}$ if you drop it without the larger ball. NB: If you would use three balls instead of two, the third ball would have a velocity of $7\vec{v}$ using the same reasoning as above.

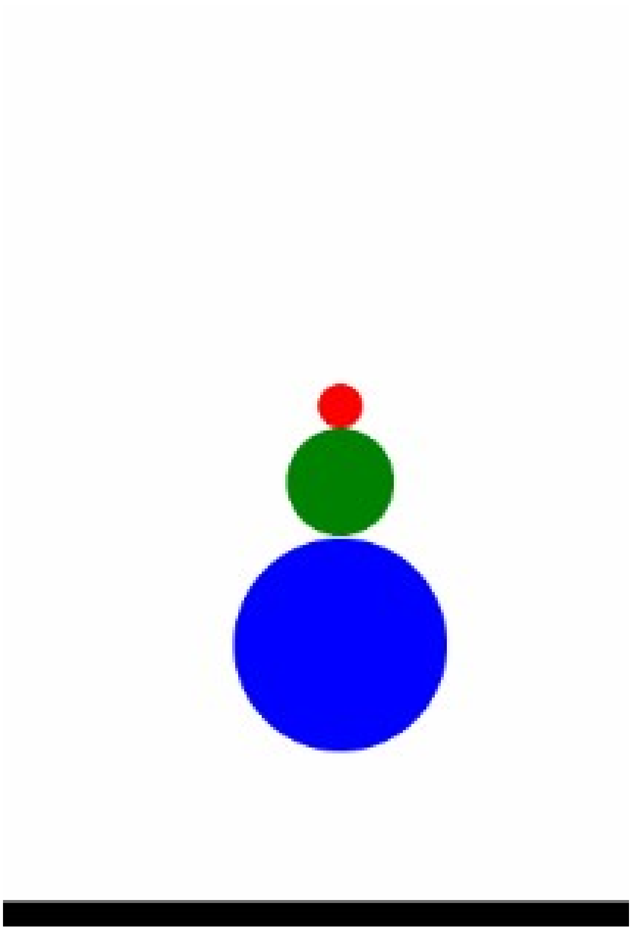


Figure 2.218: Bouncing of three (super)balls.

How much higher does the smaller ball fly with velocity $3\vec{v}$ compared to $\vec{v}$?

Answer
We equate the kinetic energy when the ball is just reflected with the potential energy when the ball reached it maximal height before falling back.

$$\frac{1}{2}mv^2 = mgh \Rightarrow h = \frac{v^2}{2g} \tag{2.317}$$

Therefore the ball with $3v$ flies 9 times higher.

What is very fishy about this whole outcome?
In situation **1** the kinetic energy is $\frac{1}{2}m_s v^2 + \frac{1}{2}m_\ell v^2$, but in situation **3b** it is $\frac{1}{2}m_s(3v)^2 + \frac{1}{2}m_\ell v^2$ while the potential energy is zero in both cases. This clearly does not add up! But energy must be conserved under all circumstances!

The conclusion is, that we did make an approximation and did not solve the energy and momentum conservation equations for elastic collisions. Even for the case $M \gg m$ there is some momentum transfer. If you solve for the velocity of m after the collision with M, you obtain

$$v' = \frac{\frac{m}{M} - 1}{\frac{m}{M} + 1} v \tag{2.318}$$

For $M \gg m$ you indeed see $v' = -v$. Thus the smaller ball will have a smaller velocity than reasoned above *and* the larger ball with also have a smaller velocity (in the experiment you can clearly notice that it does not fly as high as when it drops without the small ball on top). In real life, the balls also deform which makes the collision inelastic.

In a later chapter we will deal with collisions and pay attention to this limit $M \gg m$ in much more detail.

2.5.5.3 Examples

Example: 8.1

Consider yourself biking at a constant velocity on an unlikely day with zero wind. Still, you experience a frictional force from the air, with the following observation: the faster you bike, the larger this force. An experimentalist is trying to measure the friction force of the air and relate it to your velocity. She finds that, by and large, these forces turn out to scale with the square of your velocity v_b

$$F_f \propto v_b^2 \tag{2.319}$$

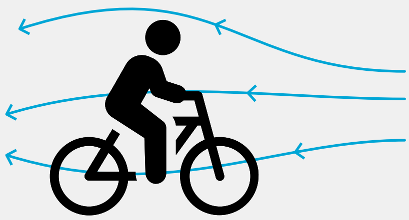


Figure 2.219: Air resistance on cyclist.

Understanding the Galilean transformation, you immediately see that this can't be correct. In your frame of reference, your velocity is zero. And thus, the friction force would be zero. But that cannot be true: both observers should see the same forces. What you see is that the air is blowing at a speed $v_{air} - v_b$ past you. And indeed, the faster you bike, according to the experimentalist, the faster you see the air moving past you: velocity is relative.

You quickly realize that a proper description of the air friction must depend on the relative velocity between you and the air. *Relative* velocities are invariant under Galilean transformation:

$$F_f \propto (v_b - v_{air})^2 \tag{2.320}$$

Example: 8.2

Riding a bike while it rains. You have done this hundreds of times. Your front gets soaked, while the backside of your coat stays dry. Or if you have a passenger on your carrier he/she will not get wet, while you take all the water. From a GT to the reference frame of the biker it is obvious why this is the case. The rain is not falling straight from the sky, but at an angle towards him.

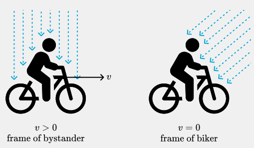


Figure 2.220: Riding a bike in the rain.

NB: For Dutch bikers you have had this experiences with head wind and rain all your life.

2.5.5.4 Demo

A ball is bouncing at a wall. The mass of the wall is much greater than that of the ball. So, acceleration of the wall or changes in momentum of the wall can be ignored.

On the left side, we see this from the perspective of an observer, S, standing next to the wall. The right side shows what observer S', who is traveling with the ball as it moves towards the wall, sees. Notice, that both S and S' are inertial observers. That is, they keep their velocity and are no part of the collision.

What would Galilei say?

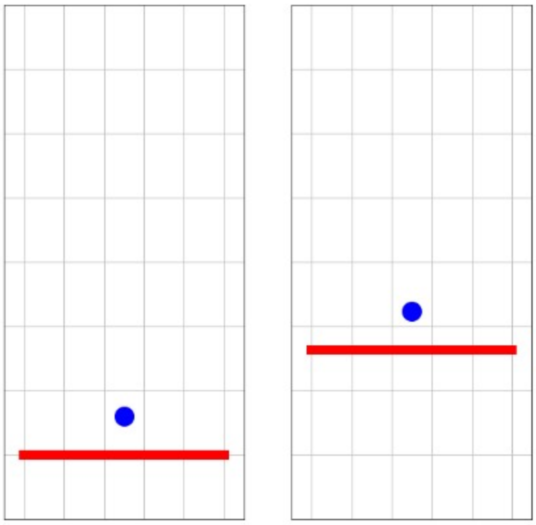
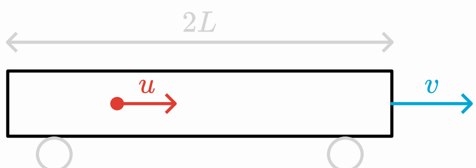


Figure 2.221: Ball bouncing at a wall.

Exercise 2.222: 🌶

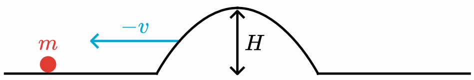
A train is passing a station at a constant velocity V. At the platform, an observer S sees that in the middle of the train (train length $2L$), at $t = 0$ an object is released with a constant velocity u. The object moves towards the back of the train and, at some point in time, will hit the back.



Inside the train, observer S' sees the same phenomenon. Show that both find the same time for the object hitting the back of the train.

Exercise 2.224: 🌶️

A point particle of mass m is sitting on a horizontal frictionless table. Gravity is acting in the vertical downward direction.
According to your observation, m has zero velocity. But you see the table moving at a velocity $-v$ in the negative x-direction. The table doesn't stay flat, but has a bump of height H. What will happen to m?



Exercise 2.226: 🌶️

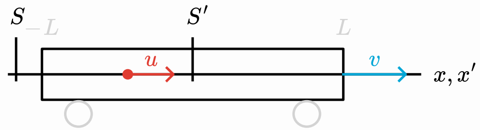
Finally, it is winter. And this time, there is lots of fresh snow! You get engaged in a great snowball fight. Your opponent has run out of 'ammunition' and runs away. She is at a distance $L = 2$m when she starts running at a speed of 5m/s. You throw your last snowball at her at a speed of 10m/s.
Determine when and where the snowball hits her. Do that three times:

- Your perspective;
- Your opponent's perspective;
- The snowballs perspective.

Next, use the Galilean Transformation and show that you could have used your perspective and GT to find the data for the other two perspectives.

Solution 2.227: Solution to Exercise 1

First we make a new sketch, now showing the two observers S and S' and their axis. We have made the velocity of the object red, the color of S. And we have given the coordinates of the front and back of the train in grey as these are specified according to S'. We do this, as it is is crucial to realize that we have 'mixed' information.



The velocity of the object is u according to S. The observer in the train, S', sees a different velocity.
The observer in the train will denote the position of the front of the train by ${x_f}' = L$ and of the back ${x_b}' = -L$. Both are, according to S', fixed values. But S will see that differently.

According to S', the object moves with velocity $u' = u - V$. Note that this is a negative value, otherwise the object will not hit the back of the train.

S' will describe the trajectory of the object by: $x'(t) = {x'}_0 + u't$ with ${x'}_0 = 0$. Thus, the object will hit the back of the train at:

$$x'(T') = -L \rightarrow u'T' = -L \rightarrow T' = \frac{L}{-u'} \tag{2.321}$$

What does S observe? It will write for the trajectory of the object $x_o(t) = ut$ (where we used that the object was released in the middle of the train at $t = 0$ and both observers chose that as their origin).
According to S also the back of the train is moving. It follows a trajectory $x_b = -L + Vt$, since at $t = 0$ the back of the train was at position $x = -L$ according to S. The two will collide when

$$x_o(T) = x_b(T) \rightarrow uT = -L + VT \rightarrow T = \frac{L}{V - u} \tag{2.322}$$

Hence we have T and T' as times of collision. But we already found $u' = u - V$. If we substitute this in T' we get

$$T' = \frac{L}{-u'} = \frac{L}{V - u} = T \tag{2.323}$$

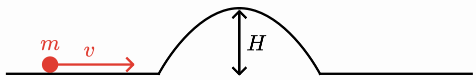
Thus, indeed both observers see the collision at the same moment.

Sneak Preview: much to our surprise, when we enter the world of Special Relativity, this will no longer be the case!

Solution 2.229: Solution to Exercise 2

The particle will 'collide' with the bump. This might cause the particle to start moving to the left. How to analyse this situation?

Perhaps it is easier when we view this from the point of view of an observer moving with the table.



Now we have a situation of a particle moving over a friction less table with velocity v. If we use conservation of energy, we can write down:

$$\frac{1}{2}mu^2 + mgh = E_0 = \frac{1}{2}mv^2 \tag{2.324}$$

where we have taken h as the height above the table and denote the velocity of m at some point by u. The initial height is zero and the initial velocity is v.

So, if the initial velocity is such that $\frac{1}{2}mv^2 > mgH$, the particle will go over the bump and come back to height $h = 0$. It will thus pass the bump and then continue moving with velocity v. For the original observer this means: the bump will pass the particle and after passing the particle is again laying still (but not at the same position!).

If, on the other hand v is such that $\frac{1}{2}mv^2 < mgH$, the particle will not reach the top of the bump: it has insufficient kinetic energy. In stead it will stop at some height $h^* = \frac{v^2}{2g}$ and then fall of the bump again. It will continue with velocity $-v$ at the flat part of the table. To the original observer this means that m first climbs the bump and returns to get a velocity $-2v$ on the flat part of the table.

The final possibility is $\frac{1}{2}mv^2 = mgH$. In that case the particle will exactly reach the top of the bump and stop there.

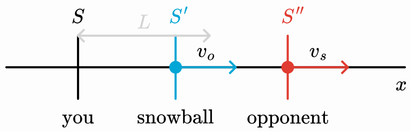
N.B. We have assumed that the bump is not too steep, because in such a case the particle will have a real collision with the bump. Think, for instance, of the bump as a sudden step. Then no matter how fast the particle is moving, it will not end up on the step, but bounce back.

2.5.5.5 Exercises

2.5.5.6 Answers

Solution 2.231: Solution to Exercise 3

First, a sketch:



It is a 1-dimensional problem, so an x-axis will do. We denote the velocity of your opponent (as seen by you) by v_o and of the snowball v_s. The inertial system of you is S and you are sitting in the origin $\mathcal{O}$. Similarly, you opponents inertial system is S' with origin $\mathcal{O}'$ and finally the snowball has inertial system $S"$ and the snowball sits in the origin $\mathcal{O}"$.

1. Your perspective

$$x_s(t) = v_s t \tag{2.325}$$

$$x_o(t) = L + v_o t \tag{2.326}$$

require: $x_s(t^*) = x_o(t^*)$

$$\rightarrow t^* = \frac{L}{v_s - v_o} = 0.4s \rightarrow x^* = v_s t^* = 4m \tag{2.327}$$

1. Your opponent's perspective

$$v'_s = v_s - v_0 = 5m/s \tag{2.328}$$

require: $x'_s(t'^*) = 0$ since S' is in $x' = 0$. Thus

$$x'_s(t'^*) = -L + v'_s t'^* = 0 \rightarrow t'^* = \frac{L}{v'_s} = 0.4 \tag{2.329}$$

Same time of course. Position: your opponent concludes she is not moving and this she is hit at $x' = 0$.

3. The snowballs perspective.

According to the snowball $v"_o = v_o - v_s = -5m/s$. Thus,

$$x"_o = L + v"_o t \tag{2.330}$$

require: $x"_o(t"^*) = 0$

$$x"_o(t"^*) = L + v"_s t"^* \rightarrow t"^* = -\frac{L}{v"_o} = 0.4s \tag{2.331}$$

And, again the snowball will conclude that it all happened in its origin.

Galilean Transformation
We now have three different time/place coordinates for the event 'snowball hits opponent'.

$$\begin{aligned} S &: (x_h, t_h) = (4m, 0.4s) \\ S' &: (x'_h, t'_h) = (0m, 0.4s) \\ S" &: (x"_h, t"_h) = (0m, 0.4s) \end{aligned} \tag{2.332}$$

We could have found this directly from a GT.
a. from S to S': we need to take into account that at $t = 0$ the origins do not coincide. Instead $\mathcal{O}'$ is shifted over a distance L w.r.t. $\mathcal{O}$

$$\begin{aligned} x' &= x - L - v_o t \\ t' &= t \end{aligned} \tag{2.333}$$

Thus: $x'_h = x_h - L - v_o t_h = 0$ and we get indeed $(x'_h, t'_h) = (0\text{m}, 0.4\text{s})$

b. We do a similar exercise for S to $S"$:

2.6 Oscillations

Updated: 04 feb 2026

2.6.1 Periodic Motion

There are many, many examples of periodic systems. We see them in physics, like the orbit of planets around their star. We find them in biology (like the predator-prey systems), in chemistry (oscillating reactions like the Belousov-Zhabotinsky reaction), and in economics (like demand-supply fluctuations). They show up in daily life: the day-night rhythm, the tides, children on a swing, your heart-beat. Periodic motions are by definition motions that repeat themselves after a fixed period of time, usually called 'the period'.

A specific class of periodic motion is known as oscillatory motion, or simply oscillations. All oscillations are periodic, but not all periodic motions are oscillations. An oscillation involves movement back and forth around an equilibrium position. It is typically caused by a restoring force: a force that acts to return the system to equilibrium (in case of the mass spring system: $\vec{F} = -k\vec{u}$). However, due to inertia, the system overshoots this position. The restoring force then reverses direction, pushing the system back again, leading to continued oscillation.

A few simple examples will illustrate the above.

2.6.1.1 The merry-go-round

The merry-go-round (Figure 2) is a periodic motion, but not an oscillation. The seats go round in a circular, periodic motion but there is no back & forth. This is in contrast to a swing. That is also a periodic motion, but it has the back and forth as well as a restoring force, which in this case is gravity.

Figure 2.234: Spinning carousel. By Oxana Mayer, from Wikimedia Commons, licensed under CC BY-SA 2.0.

2.6.1.2 Rabbits and Foxes

As an example of a dynamic system that is periodic, we will take a look at the so-called predator-prey systems. These are well-known in biology and provide an interesting case. The idea is simple: the populations of rabbits growth as they multiply quickly. The idea in the prey-predator model is that growth rate is proportional to the population itself. For the rabbits that means that the derivative of the population of rabbits (with respect to time) is positive. If there are no foxes, the rabbit population will grow exponentially. Of course, in the real world that doesn't happen as sooner or later, the rabbits will ran out of food, resulting in starvation. However, we will assume here, that food is not limiting: but the number of foxes is. They stop the rabbit population from unbounded increasing. The more rabbits there are, the easier the foxes find food and the more foxes will survive childhood. A simple model reads as follows:

$$\begin{aligned} \frac{dr}{dt} &= \lambda_r r - \mu_r r \cdot f \\ \frac{df}{dt} &= -\lambda_f f + \mu_f r \cdot f \end{aligned} \tag{2.335}$$

here r and f represent the rabbit and fox population, resp. λ_r is the growth rate of the rabbits: the more rabbits, the larger the offspring. The higher λ_r the more babies per rabbit. μ_r, on the other hand, represents the effectiveness of the hunting foxes: the larger this value the more rabbits they kill. Of course: more rabbits, but also more foxes also means more kills. Similar arguments apply to λ_f and μ_f. Note that the term with λ_f carries a negative sign: the net increase of the fox population is negative if there is insufficient food, that is, by itself more foxes die then that are born if there is no food.

This is clearly a coupled and dynamic system. It is non-linear due to the product $r \cdot f$, making it much more difficult to solve analytically then linear versions. In literature, this kind of system is known as Lotka-Volterra or prey-predator models. Below is a plot of the numerical solution of the rabbit and fox population (for $\left(\lambda_r, \mu_r, \lambda_f, mu_f\right) = (0.2, 0.03, 0.1, 0.01)$ and initial conditions (r_0, f_0) = (80, 2)).

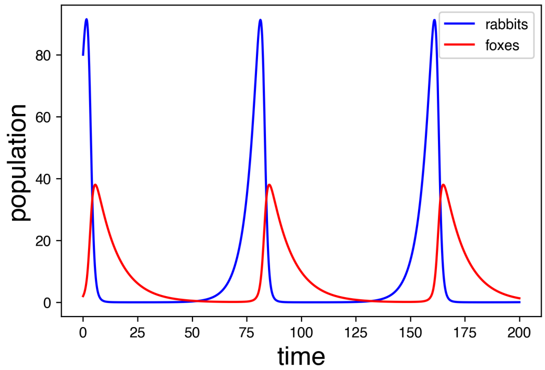


Figure 2.236: Periodic time evolution of the population of rabbits and foxes.

The solution is periodic. This can be illustrated better by plotting f against r. The animation below shows this (this kind of plot is called a phase plot).

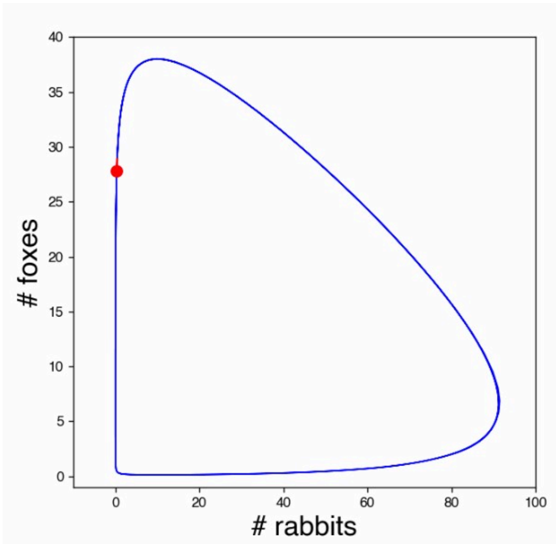


Figure 2.237: Phase plot of the rabbit-fox prey-predator model. The red dot shows the population at different times. Note that the number of rabbits quickly increases when there are very few foxes. However, at some point the number of foxes also goes up and soon the start reducing the rabbits, while increasing in numbers themselves. That is not sustainable and when the number of rabbits is brought down substantially, also the number of foxes decreases, until both are almost extinct and the cycle repeats.

2.6.1.3 Wilberforce Oscillator

As a third example we look at the Wilberforce pendulum. This is a spring, suspended vertically, to which a weight is fixed at the free end. The weight can go up and down but also rotate in a horizontal plane. A sketch is given below.

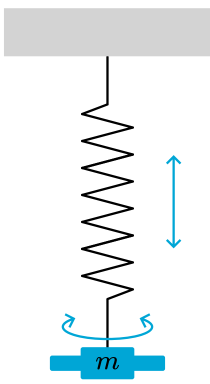


Figure 2.238: Wilberforce pendulum.

Imagine that we pull m a little down and let go. The spring will try to restore the position of the mass to the equilibrium position it was in prior to us pulling m down. Consequently, m will start oscillating in the vertical direction. However, something peculiar happens: the mass m also starts to rotate (around the vertical axis). And also this rotation turns out to be a back and forth oscillation. But that is not all: the two oscillations are coupled: they feed each other. If the vertical oscillation is at a maximum amplitude, the rotational motion is almost zero and vice-versa.

Figure 2.239: A Wilbertforce pendulum made by first year physics students

The system can be modeled with simple means. We will just postulate them. Later on, we will see where the terms come from.
First, we note that the mass has kinetic energy, in two forms: due to the vertical motion ($\frac{1}{2}m\dot{z}^2$) and due to the rotational motion ($\frac{1}{2}I\dot{\theta}^2$). Don't worry about the exact meaning for now.
Second, the mass has potential energy. We will ignore gravity (we could for instance do the experiment in the International Space Station, ISS). A potential energy is associated with the vertical motion and is the spring energy: $V_z = \frac{1}{2}kz^2$, with z the vertical position of the mass with respect to the equilibrium position, which we took as $z = 0$. k is the spring constant and represents the strength of the spring. We will come back to this later.
Then, we have potential energy associated with the rotation: $V_\theta = \frac{1}{2}\delta\theta^2$. θ represent the rotation angle, where we have taken $\theta = 0$ in the equilibrium position. δ is the torsional spring constant: it represents how strongly the spring tries to push back against rotation.
Finally, the vertical position and the rotation influence each other. That can be understood by realizing that if you shorten the spring, the spring material has to go somewhere. It cannot only change its vertical length as that would mean that the total length of the spring would reduce. But that would compress the spring material and that is not possible for solid material (unless you apply incredibly large forces). The spring just increases its number of windings a bit. But that implies rotation. Similarly, if we only rotate the spring, it will try to

adjust its length. As a consequence, there is also a potential energy involved in the influencing of z and θ of each other. It can be modeled as $V_{z\theta} = \varepsilon z\theta$.

If we ignore friction, then we have a system that can be described in terms of energy:

$$\frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}kz^2 + \frac{1}{2}\delta\theta^2 + \varepsilon z\theta = E_0 \tag{2.336}$$

From this, we can find 'N2', the equation of motion:

$$\begin{aligned} m\ddot{z} &= -kz - \varepsilon\theta \\ I\ddot{\theta} &= -\delta\theta - \varepsilon z \end{aligned} \tag{2.337}$$

Don't worry, if you don't follow this. The point here is, that we have a coupled system of two oscillators. This can be solved numerically.

We could use a simple numerical scheme like we have employed in Chapter 3. In the figure below $z(t)$ and $\theta(t)$ are shown using such a simple numerical scheme.

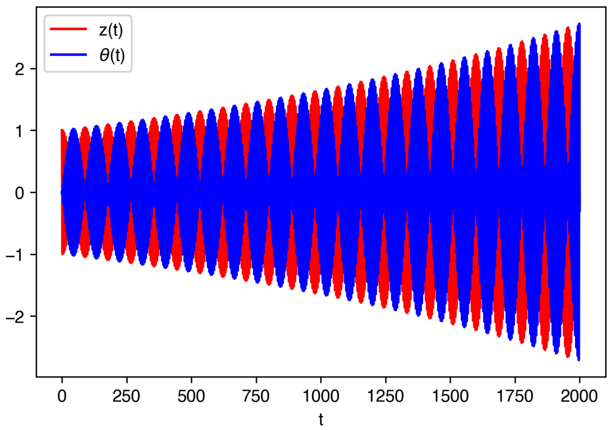


Figure 2.240: Numerical solution of the Wilberforce pendulum using a (too) simple numerical method.

We indeed see the oscillating motion and that the vertical oscillation changes over to rotation and back again.

But there is something really disturbing: the amplitude of our oscillation is increasing and it seems to do so for every cycle. That cannot be true: It violates energy conservation. What did we do wrong? Well, our numerical method is just not good enough. If we use again a higher order method, we obtain the results in the figure below.

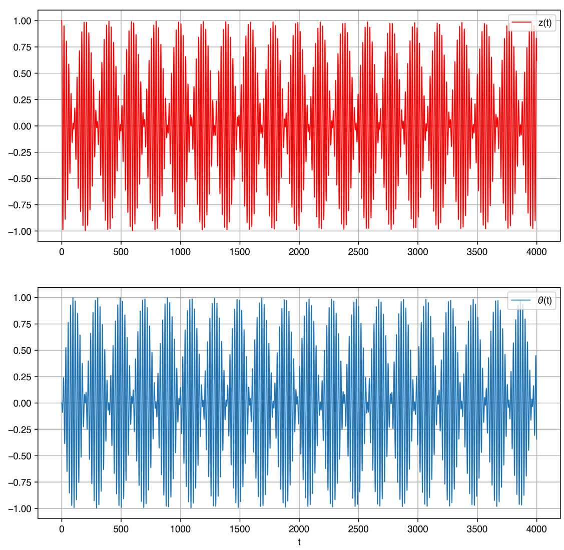


Figure 2.241: Numerical solution of the Wilberforce pendulum using a higher-order numerical method.

Now the amplitude of the oscillations stays nicely constant, obeying conservation of energy.

In the figure below a small animation can be seen: the marker in both graphs shows z and θ at the same time instant.

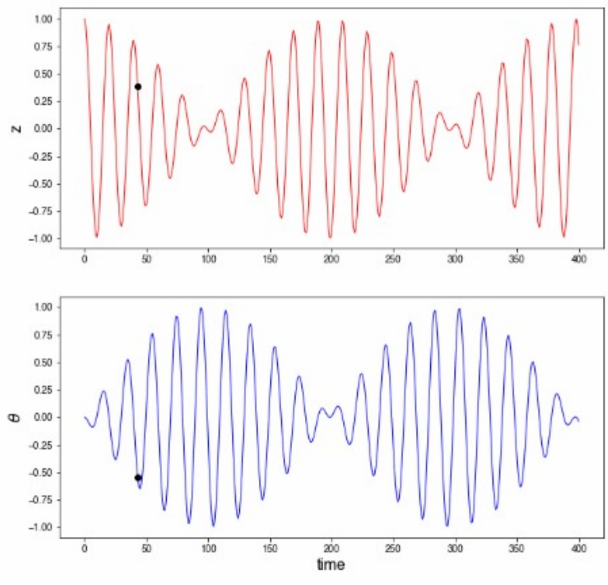


Figure 2.242: Animation of the Wilberforce pendulum using a higher-order numerical method.

The Wilberforce pendulum is clearly periodic. Moreover, it is an oscillation as there is back and forth motion around an equilibrium.

But, it does give us a **big warning**: (numerical) solutions always have to be **assessed** against the features and principles of the problem at hand. In this case, our first numerical solution could not be right: **it violated energy conservation**. We were able, right from the start, to formulate the problem in terms of energy. Since we only had kinetic energy and potential energy we **knew up front** that the motion must be bounded!
That is why, we need a thorough understanding of physics. It is not sufficient to have the equations and put them in a 'solver'. It is the job of a physicist to understand and assess models, outcomes, etc against the laws of physics. Hence, we will dive into oscillations, starting from the beginning.

2.6.2 Harmonic Oscillation - archetype: Mass-Spring

The archetype of an oscillation is the mass-spring system. It is the simplest version (simpler than the pendulum as we will see). And it can be recognized in many systems. We consider the following: a mass is attached to a spring. The other end of the spring is fixed. The mass can only move in one direction: the x-direction. The spring has a natural or rest length l_0. That is the length of the spring if no force is acting on it. If we pull the spring, it will exert a force that is proportional to the increase in length. Moreover, it is pointing in the direction opposite to the lengthening. In formula:

$$F_v = -k(l - l_0) = -k\Delta l \tag{2.338}$$

This is shown in the figure below.

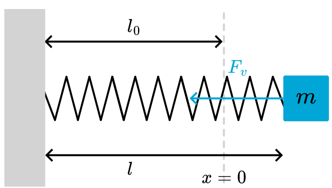


Figure 2.243: Mass-spring system: archetype of a (harmonic) oscillation.

The response of the spring is to exert a force on m proportional to its elongation (which may be negative, i.e. the spring is compressed). It is clearly a restoring force: no matter what we do pulling or pushing, the spring will always counteract.

It is not difficult to set up N2 for the mass-spring. There is only one force and the system is 1-dimensional. If we define the origin at the position of the mass when the spring is at its rest length, then Δl - the elongation of the spring - becomes x, the coordinate of the mass m. Thus N2 reads as:

$$m\ddot{x} = -kx \tag{2.339}$$

Or

$$m\ddot{x} + kx = 0 \tag{2.340}$$

To solve this, we need two initial condition. Let's take $t = 0 : x(0) = x_0, v(0) = 0$. We need to find a function $x(t)$ that upon differentiating twice it spits itself back but with an opposite sign. We do know two functions that do so: $x(t) = \sin(\omega_0 t)$ and $x(t) = \cos(\omega_0 t)$. Thus, the general solution of the above equation is known.

Harmonic Oscillator:

$$m\ddot{x} + kx = 0 \Leftrightarrow x(t) = A \sin \omega_0 t + B \cos \omega_0 t \tag{2.341}$$

If we insert the solution, we find

$$\omega_0^2 = \frac{k}{m} \tag{2.342}$$

This is called the natural frequency of the oscillator. Note, that it does not depend on the initial conditions. No matter what, the mass will always oscillate with this frequency.

It does make sense that the frequency is inversely proportional to m: we expect a heavy object will respond slow to a force. Similarly, if the spring is strong, that is has a high spring constant k, it will move the mass around quickly.

If we substitute the initial condition, we can completely solve the motion of the mass:

$$\begin{aligned} &m\ddot{x} + kx = 0 \Rightarrow x(t) = A \sin \omega_0 t + B \cos \omega_0 t \Rightarrow \\ &\Rightarrow x(t) = \Delta x \cos \sqrt{\frac{k}{m}} t \end{aligned} \tag{2.343}$$

A system is called a harmonic oscillator if and only if it obeys $m\ddot{x} + kx = 0$. You will find them in almost every branch of science and engineering. The reason why will become apparent in a moment.

2.6.2.1 Potential energy of a spring

In the above, we have formulated the mass-spring system in terms of Newton's second law. We can, however, also cast it in the form of energy. The force of the spring is conservative. We can easily prove this by finding the associated potential energy: $F_v = -\frac{dV}{dx}$.
Since $F_v = -kx$ we need to find a function $V(x)$ that satisfies $\frac{dV}{dx} = kx$. Let's do it:

$$\frac{dV}{dx} = kx \Rightarrow V(x) = \frac{1}{2}kx^2 + C \tag{2.344}$$

We have the freedom to decide ourselves where we want the potential energy to be zero.
Note: V is quadratic.
It does make sense, to set the minimum of the potential energy such that if the mass is at the equilibrium position, the potential energy is zero, that is - take $C = 0$:

$$V(x) = \frac{1}{2}kx^2 \tag{2.345}$$

Thus the mass-spring system can also be described by

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E_0 \tag{2.346}$$

So, an other way of stating what a harmonic oscillator is: it is a system that obeys the above energy equation.

2.6.3 Behavior around an equilibrium point and harmonic oscillators

Now we will go back to paragraph 5.5.1, where we discussed the Taylor series expansion of the function $f(x)$:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \mathcal{O}(x^3) \tag{2.347}$$

We will apply it to a potential energy $V(x)$ of some system. We assume that the system has a stable equilibrium point at $x = x_0$, that is $\left[\frac{dV}{dx}\right]_{x=x_0} = 0$ and $\left[\frac{d^2V}{dx^2}\right]_{x=x_0} > 0$.
Thus, we can expand the potential as follows:

$$V(x) \approx V(x_0) + \underbrace{\frac{1}{2}\left[\frac{d^2V}{dx^2}\right]_{x=x_0}}_{=k}(x - x_0)^2 + \mathcal{O}\left[(x - x_0)^3\right] \tag{2.348}$$

If we plug this in, in the energy equation and cut off after the quadratic term, we find

$$\frac{1}{2}mv^2 + V(x_0) + \underbrace{\frac{1}{2}\left[\frac{d^2V}{dx^2}\right]_{x=x_0}}_{=k}(x - x_0)^2 = E_0 \tag{2.349}$$

or shortened by the abbreviation $\left[\frac{d^2V}{dx^2}\right]_{x=x_0} = k$

$$\frac{1}{2}mv^2 + V(x_0) + \frac{1}{2}k(x - x_0)^2 = E_0 \tag{2.350}$$

Move the constant $V(x_0)$ to the right hand side and change coordinate $s \equiv x - x_0 \rightarrow \dot{s} = \dot{x} = v$. This gives us:

$$\frac{1}{2}m\dot{s}^2 + \frac{1}{2}ks^2 = C \tag{2.351}$$

The harmonic oscillator!!! No wonder we find harmonic oscillators 'everywhere'. Any system that has a stable equilibrium point with a positive second derivative of its potential will start to oscillated as a harmonic one if we push it a little bit out of its equilibrium position. Doesn't matter how $V(x)$ exactly is. It doesn't have to be quadratic in x. But it will be pretty close to that, if we stay close enough to the equilibrium point. Hence, any small natural kick, any small amount noise will push a system out of its stable equilibrium point into an harmonic oscillating motion with a given, natural frequency given by $\omega_0^2 = \frac{\left[\frac{d^2V}{dx^2}\right]_{x=x_0}}{m}$.

2.6.4 Examples of Harmonic Oscillators

2.6.4.1 Torsion Pendulum

We take a straight metal wire. Suspend one end at the ceiling and attach a disc of radius R and mass m at the other end.

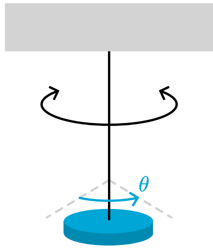


Figure 2.244: Torsion Pendulum.

The disk can rotate about a vertical axis. We call the rotation angle θ. The equilibrium position is $\theta = 0$. If we rotate the disc over a small angle, the wire will resist and apply a torque τ on the disc trying to rotate the disc back to its equilibrium position, for which the torque, obviously is zero.

For small angles, the torque is proportional to the rotation angle and -of course -working in the direction opposite of the rotated angle. We can set up an angular momentum equation and find that it reads as:

$$I\frac{d^2\theta}{dt^2} = -k_t\theta \tag{2.352}$$

In this equation, $I = \frac{1}{2}mR^2$ is the moment of inertia of the disc and k_t is the torsion constant of the wire. Don't worry about the exact meaning of the terms in the equation. For now, we focus on the equation itself:

$$I\frac{d^2\theta}{dt^2} + k_t\theta = 0 \Rightarrow \theta(t) = A\sin\omega_0 t + B\cos\omega_0 t \tag{2.353}$$

The torsion pendulum is a harmonic oscillator, $\omega_0^2 = \frac{k_t}{I}$, completely analogous to the archetype, mass-spring. Obviously, we thus can also write this in terms of energy:

$$\frac{1}{2}I\omega^2 + \frac{1}{2}k_t\theta^2 = E_0 \tag{2.354}$$

with $\omega \equiv \frac{d\theta}{dt}$, the angular velocity.

2.6.4.2 L-C circuit

In Electronics alternating current (AC) circuits are building blocks of many complex systems. One of these is the L-C circuit, in which an inductor, L, and a capacitor, C, are in series coupled. See Figure 13.

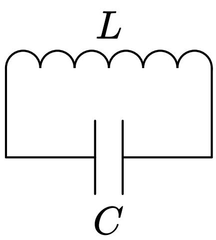


Figure 2.245: L-C circuit.

Intermezzo: Kichhoff's Laws

In dealing with electronic circuits, there are two helpful laws. They carry the name of the German Physicist Gustav Kirchhoff, who wrote them down in 1845.The first law is 'current law'. Picture a node in an electronic circuit, that is a point where a number of elements are connected to each other. This may be resistors, capacitances, batteries, any element. Then the total current flowing in to the node is the same as the total current flowing out of the node. In other words: a node cannot store charge; what flows in must flow out. In a more mathematical form we write: the sum of currents in a node is zero.Currents in and out of a node must add to zero.In the example in the figure, with a resistor a capacitance and a battery, this is: The second law is the Kirchhoff's Voltage

Law. Now we go around a loop in a circuit, that is any loop you can find. We start at a certain point and after each element we write down the voltage after that element. Once we have returned to our starting point, we sum up all voltage differences that we now have (that is the voltage difference over each element). The result is zero: no matter which loop we take, the sum of voltages differences is always zero: .Voltage differences across a loop add up to zero.In the example in the figure, again with a resistor, a capacitance and a battery, this is: It is comparable with a closed loop walk in the mountains: you start at a certain point with a certain height. Then, you go up and down, up and down. But at the end of the walk: you gained no height and didn't loose it either (despite all your effort 😊).

We could charge the capacitor and then close the circuit. What would happen? The capacitor will try to discharge via the inductor. Hence a current, I, starts flowing. In response, the inductor builds up a potential difference that is directly proportional to the rate of change of the current through the inductor.

Basic electronics shows that the voltage over the capacitor is coupled to the charge, Q_C, of the capacitor according to: $V_C = \frac{Q_C}{C}$. For the inductor we have: $V_L = L\frac{dI_L}{dt}$.

According to Kirchhoff's laws the current through both elements must be the same: $I_C = I_L$ and the sum of the voltages across them must be equal to zero: $V_c + V_L = 0$. If we put everything together, we get - using $I_C = \frac{dQ_c}{dt}$:

$$\begin{aligned} V_L + V_C &= 0 \Rightarrow \\ \frac{dV_L}{dt} + \frac{dV_C}{dt} &= 0 \Rightarrow \\ L\frac{d^2I}{dt^2} + \frac{1}{C}I &= 0 \Rightarrow \\ \frac{d^2I}{dt^2} + \frac{1}{LC}I &= 0 Harmonic Oscillator!!! \end{aligned} \tag{2.355}$$

As we see, this LC-circuit will start to oscillate. In the animation below the current through the circuit and the voltage across the inductor are shown for $C = 1\mu F$ and $L = 1\mu H$.

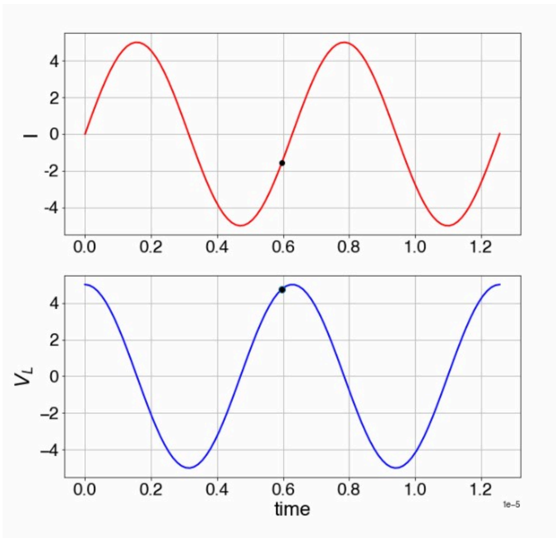


Figure 2.246: Harmonic oscillation of an LC-circuit.

2.6.4.3 Musical Instruments

Musical instruments produce sound waves. In many cases they do that via vibrations of strings, like the guitar, the violin, harp or piano. The strings of these instruments are displaced out of their equilibrium position. Due to the tension in these strings, there is a restoring force that is proportional to the displacement. Consequently, the string will start to oscillate in an harmonic way.

Not only strings, but also beams will exhibit this behavior, well-known example: a tuning fork. We will come back to waves at the end of this chapter.

2.6.5 The pendulum

Another example of oscillatory motion is the pendulum. In it's most simple form it is a point-mass m, attached to a massless rod of length L. The rod is fixed to a pivotal point that allows it to swing freely.

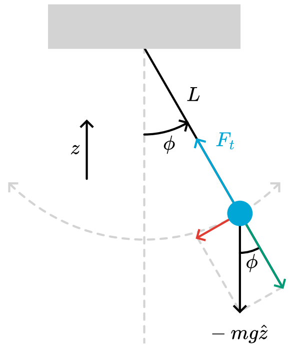


Figure 2.247: Sketch of a pendulum.

On the mass, gravity is acting vertically downwards. Also the rod exerts a force on the mass. This force is always parallel to the rod and points to the pivotal point. It is the response of the rod to the component of gravity parallel to the rod (the dark blue arrow in Figure 17). It is good to realize, that this force makes sure that the distance from m to the pivotal point is always L. In other words, this force is a consequence of the fixed length L of the rod. It is the physics translation of the constraint: L is constant.

2.6.5.1 N2 for the pendulum: Equation of motion via N2

We will set up Newton's second Law for m.

$$m\frac{d\vec{v}}{dt} = -mg\hat{z} + \vec{F}_t \tag{2.356}$$

As stated above, the blue, parallel part of gravity is balanced by a tensional force in the rod. So, we don't need to worry about motion of m parallel to the rod. That leaves us with the direction perpendicular of the rod. In that direction only the red arrow works on m.

In the other direction only the red, perpendicular component of gravity acts on m. This component is equal to $-mg\sin\phi$. The velocity component in this direction is $v = L\frac{d\phi}{dt}$. Thus we get:

$$mL\frac{d^2\phi}{dt^2} = -mg\sin\phi \tag{2.357}$$

Or rewritten

$$mL\frac{d^2\phi}{dt^2} + mg\sin\phi = 0 \tag{2.358}$$

We do know from experience that the pendulum will swing back and forth in a periodic way. However, as we see from the above equation of motion: it is not a harmonic oscillator. The term with the sine prevents that.

But for small values of the angle ϕ, that is for small oscillations around the stable equilibrium $\phi_{eq} = 0$, we can approximate the sinus via a Taylor series and write:

$$\phi \ll 1 \Rightarrow \sin\phi \approx \sin 0 + \frac{1}{1!}\cos 0\,\phi - \frac{1}{2!}\sin 0\,\phi^2 + \ldots \\ \approx \phi \tag{2.359}$$

Thus within this approximation we can write for the equation of motion of the pendulum:

$$mL\frac{d^2\phi}{dt^2} + mg\phi = 0 \Rightarrow \frac{d^2\phi}{dt^2} + \frac{g}{L}\phi = 0 \tag{2.360}$$

and that describes a harmonic oscillator.

We conclude that for small amplitudes of the oscillation, the pendulum is an harmonic oscillator and swings in a sine or cosine way back and forth. Moreover, the oscillation has a frequency

$$\omega_{pendulum} = \sqrt{\frac{g}{L}} \tag{2.361}$$

Further, note that under this assumption, the period of the pendulum does not depend on the amplitude of the oscillation. That was already noted by Galileo Galilei.

2.6.5.2 N2 for the pendulum: Equation of motion via Angular Momentum

Before we continue with the analysis of the pendulum, we will derive the equation of motion also via angular momentum considerations. On m gravity exerts a torque: $\vec{\tau} = \vec{r} \times \vec{F}_g$ It has a magnitude $-Lmg\sin\phi$ and points into the screen. The angular momentum of m is given by $\vec{L} = \vec{r} \times \vec{p}$. This has magnitude $mL^2\frac{d\phi}{dt}$ and also points into the screen.

Thus N2 for angular momentum gives us:

$$\frac{d\vec{L}}{dt} = \vec{\tau} \Rightarrow mL^2\frac{d^2\phi}{dt^2} = -Lmg\sin\phi \tag{2.362}$$

Thus, angular momentum leads to the same equation of motion.

2.6.5.3 The Pendulum via energy conservation

Alternatively, we can also use energy conservation to derive the equation governing the motion of the pendulum. There are, as discussed above, two forces acting on m. The first one is gravity, which is a conservative force with associated potential energy. We can write for this case $V_g = mgz$, taking $V_g(z = 0) = 0$.

The second one is the force from the rod. But this one always acts perpendicular to the motion of m. Hence, it does not do any work and, thus, we don't need to worry about an associated potential.

We conclude that for the pendulum it holds that:

$$\frac{1}{2}mv^2 + mgz = E_0 \tag{2.363}$$

To solve this, we change from z to ϕ. z is, in terms of ϕ: $L - L\cos\phi$, see Figure 18.

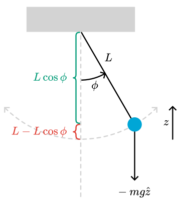


Figure 2.248: Potential energy of a pendulum.

Thus, our energy equation reads as:

$$\frac{1}{2}mv_\phi^2 + mgL(1 - \cos\phi) = E_0 \tag{2.364}$$

or

$$\frac{1}{2}mv_\phi^2 - mgL\cos\phi = E_0 - mgL \tag{2.365}$$

Take the time-derivative and use $v_\phi = L\frac{d\phi}{dt}$ and we get

$$\begin{aligned} mv_\phi\frac{dv_\phi}{dt} + mgL\sin\phi\frac{d\phi}{dt} = 0 \Rightarrow \\ mL\frac{d\phi}{dt}\frac{d}{dt}\left(L\frac{d\phi}{dt}\right) + mgL\sin\phi\frac{d\phi}{dt} = 0 \Rightarrow \\ \frac{d^2\phi}{dt^2} + \frac{g}{L}\sin\phi = 0 \end{aligned} \tag{2.366}$$

And we have recovered the same equation of motion.

2.6.5.4 Pendulum for not so small angles

In the above we have frequently used the approximation $\sin\phi \approx \phi$ for $\phi \ll 1$. What about the general case? Then we need to solve

$$\begin{aligned} \frac{d^2\phi}{dt^2} + \frac{g}{L}\sin\phi = 0 \\ with i.c. \phi(0) = \phi_0 and \frac{d\phi}{dt} = \dot{\phi}_0 \end{aligned} \tag{2.367}$$

This equation is much more difficult to solve analytically and we will, therefore, use a numerical approach here. The animation below compares the motion of the pendulum numerically simulated to that of the pendulum when using the small amplitude approximation.

The animation shows: a green mass, that is the pendulum with a (fixed) small amplitude in the approximation $\sin\phi = \phi$. The blue one uses the same approximation even though ϕ is not small. Notice, that blue and green oscillate with exactly the same frequency. This is, of course, trivial as they obey the same harmonic oscillation equation and thus have the same frequency.

The red mass, on the other hand obeys the equation of motion of the pendulum leaving the term with $\sin\phi$. It is clear that the real pendulum (i.e. the red one) does not have the same frequency as the others. Moreover, its time trace (left part of the figure) is clearly not a true sinus.

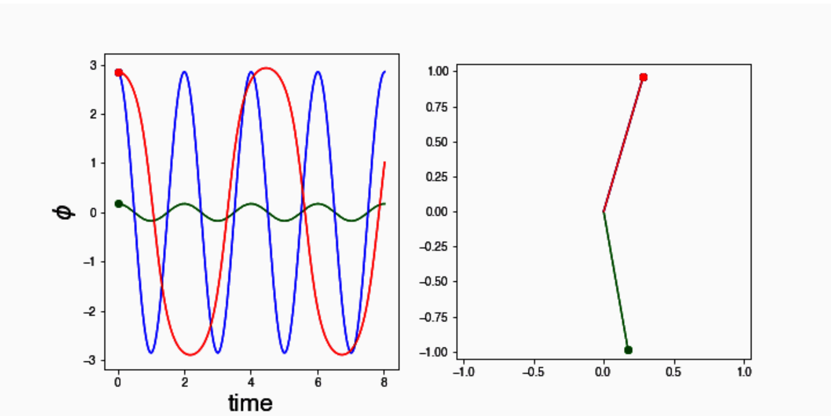


Figure 2.249: Animation of the pendulum: red is the true pendulum, blue the small angle approximation applied to a large angle case and green the small angle approximation for a small angle.

In the widget below, you can vary the initial angle and observe that indeed for a small angle the red mass and the other two follow the same trajectory. But if you increase the initial angle, the red mass behaves differently: it oscillates slower and the time trace of angle as a function of time is no longer sinusoidal.

> **Content missing, will be updated**
>
> app needs to be checked

```python
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation
from IPython.display import HTML

# Parameters
L = 1.5           # Length pendulum (scaled)
g = 9.81          # Gravity (m/s^2)
phi0 = 1.6        # Initial angle (rad)
t_stop = 10       # Total time (s)
dt = 0.05         # Time step (s)

w0 = np.sqrt(g / L)

# Time array
t = np.arange(0, t_stop, dt)

# Numerical solution of pendulum using simple Euler method
phi_num = np.zeros_like(t)
w_num = np.zeros_like(t)
phi_num[0] = phi0
w_num[0] = 0
for i in range(1, len(t)):
    w_num[i] = w_num[i-1] - (g / L) * np.sin(phi_num[i-1]) * dt
    phi_num[i] = phi_num[i-1] + w_num[i] * dt

# Harmonic oscillator approx (small angle approx)
phi_harm = phi0 * np.cos(w0 * t)

# Green pendulum: fixed small angle 0.2 rad harmonic approx
phi_fixed = 0.2 * np.cos(w0 * t)

# Setup plot
fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(12, 6))

# Left canvas: pendulum animation setup
ax1.set_xlim(-L*1.2, L*1.2)
ax1.set_ylim(-L*1.2, L*1.2)
ax1.set_aspect('equal')
ax1.set_title('Pendulum Animation')
ax1.axis('off')

rod_num, = ax1.plot([], [], 'r-', lw=3, label='Numerical Pendulum')
bob_num = ax1.plot([], [], 'ro', ms=10)[0]

rod_harm, = ax1.plot([], [], 'b-', lw=3, label='Harmonic Approx')
bob_harm = ax1.plot([], [], 'bo', ms=8)[0]

rod_fixed, = ax1.plot([], [], 'g-', lw=3, label='Fixed Small Angle')
bob_fixed = ax1.plot([], [], 'go', ms=8)[0]

ax1.legend(loc='upper right')

# Right canvas: angle vs time plots
ax2.set_xlim(0, t_stop)
ax2.set_ylim(-phi0 * 1.2, phi0 * 1.2)
ax2.set_title('Pendulum Angle vs Time')
ax2.set_xlabel('$t$ (s)')
```

```python
ax2.set_ylabel('$\\phi$ (rad)')
line_num, = ax2.plot([], [], 'r-', label='Numerical')
line_harm, = ax2.plot([], [], 'b-', label='Harmonic')
line_fixed, = ax2.plot([], [], 'g-', label='Fixed 0.2 rad')
point_num, = ax2.plot([], [], 'ro')
point_harm, = ax2.plot([], [], 'bo')
point_fixed, = ax2.plot([], [], 'go')
ax2.legend()

def init():
    for artist in [rod_num, bob_num, rod_harm, bob_harm, rod_fixed, bob_fixed,
                   line_num, line_harm, line_fixed, point_num, point_harm,
point_fixed]:
        artist.set_data([], [])
    return rod_num, bob_num, rod_harm, bob_harm, rod_fixed, bob_fixed,
line_num, line_harm, line_fixed, point_num, point_harm, point_fixed

def update(frame):
    # Current angles
    phi_n = phi_num[frame]
    phi_h = phi_harm[frame]
    phi_f = phi_fixed[frame]

    # Coordinates for bobs (pendulum rods)
    x_num = L * np.sin(phi_n)
    y_num = -L * np.cos(phi_n)

    x_harm = L * np.sin(phi_h)
    y_harm = -L * np.cos(phi_h)

    x_fixed = L * np.sin(phi_f)
    y_fixed = -L * np.cos(phi_f)

    # Update pendulum rods and bobs
    rod_num.set_data([0, x_num], [0, y_num])
    bob_num.set_data([x_num], [y_num])

    rod_harm.set_data([0, x_harm], [0, y_harm])
    bob_harm.set_data([x_harm], [y_harm])

    rod_fixed.set_data([0, x_fixed], [0, y_fixed])
    bob_fixed.set_data([x_fixed], [y_fixed])

    # Update angle vs time lines (up to current frame)
    time_so_far = t[:frame+1]
    line_num.set_data(time_so_far, phi_num[:frame+1])
    line_harm.set_data(time_so_far, phi_harm[:frame+1])
    line_fixed.set_data(time_so_far, phi_fixed[:frame+1])

    # Update current points on the plots (wrap scalars in lists)
    point_num.set_data([t[frame]], [phi_num[frame]])
    point_harm.set_data([t[frame]], [phi_harm[frame]])
    point_fixed.set_data([t[frame]], [phi_fixed[frame]])

    return rod_num, bob_num, rod_harm, bob_harm, rod_fixed, bob_fixed,
line_num, line_harm, line_fixed, point_num, point_harm, point_fixed

ani = FuncAnimation(fig, update, frames=len(t), init_func=init, blit=True,
interval=50)
HTML(ani.to_jshtml())
```

2.6.6 The damped harmonic oscillator

In the above, no friction of any form has been considered. However, in many practical cases friction will be present. For moving objects friction frequently depends on the velocity: the higher the velocity, the higher the frictional force. We will here consider the simplest version: a friction force that is directly proportional to the velocity: $F_f = -bv$ with b a positive constant. Thus, we need to add an additional force to our harmonic oscillator:

$$m\ddot{x} = -kx - b\dot{x} \tag{2.368}$$

or bringing all terms to the left hand side:

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{2.369}$$

To solve this equation, it is easier not to try to look directly for sinus and cosines, but use the complex notation.

Intermezzo: complex exponential and sin, cos

In the 18 century, the study of complex numbers, i.e. , revealed a surprising connection between the exponential function and trigonometry. It was Leonhard Euler (1707-1783) who derived:

The general solution of the (linearly) damped harmonic oscillator is:

$$\begin{gathered} m\ddot{x} + b\dot{x} + kx = 0 \Rightarrow \\ x(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t} \\ with \\ \lambda_{+,-} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \end{gathered} \tag{2.370}$$

We will investigate various cases.

D = b²-4mk *lt* **0**

In this case, the square root in λ is imaginary and we can write it as $i\sqrt{4mk - b^2}$. This gives us for the two possibilities of λ

$$\begin{aligned} \lambda_+ &= -\frac{b}{2m} + i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \\ \lambda_- &= -\frac{b}{2m} - i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \end{aligned} \tag{2.371}$$

Both have the same real part: $-\frac{b}{2m}$ showing that both solutions are damped (with the same factor!). Moreover, the imaginary parts are equal, apart from the sign which we also had in the undamped case. We will write the imaginary part as ω (that is without the subscript 0 we used for the undamped case).

So, our solution reads as:

$$\begin{gathered} x(t) = \underbrace{\left(Ae^{i\omega t} + Be^{-i\omega t}\right)}_{sinusoidal\ oscillation} \underbrace{e^{-\frac{b}{2m}t}}_{exponentialdamping} \\ with \\ \omega \equiv \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} < \sqrt{\frac{k}{m}} = \omega_0 \end{gathered} \tag{2.372}$$

Conclusion: the damped oscillator oscillates with a smaller frequency than the undamped one and it amplitude decreases over time. The later is of course to be expected due to friction: sooner or later friction has dissipated all the kinetic & potential energy.

D = b² - 4mk = 0

For this specific combination of b, k, m we see that the frequency of the oscillation is 0. In other words, the systems does not perform oscillations. Furthermore, our two values of λ are now equal. Consequently the general solution that we presented is no longer complete (we now only have one integration constant, or only one independent function if you prefer.) We need a second one and that turns out to be of the form $te^{\lambda t}$. You can verify that by substituting it in the equation of motion for the damped case.

Thus we have know:

$$\begin{aligned} D = b^2 - 4mk = 0 \Rightarrow \\ x(t) = (A + Bt)e^{-\frac{b}{2m}t} \end{aligned} \tag{2.373}$$

D = b² - 4mk > 0

Again, there is no imaginary part in λ, so no oscillations. But we do have two different values for λ and thus our original general solution is still valid:

$$x(t) = Ae^{\frac{-b+\sqrt{b^2-4mk}}{2m}t} + Be^{\frac{-b-\sqrt{b^2-4mk}}{2m}t} \tag{2.374}$$

Note that $-b + \sqrt{b^2 - 4mk} < 0$. So, both terms are decreasing to zero: the motion comes to a stop as $t \to \infty$.

Further, note that the first part (with A) has an exponent that is closer to zero than the one of the other part (with B). Thus the second part will decay faster and for sufficiently large t, the solution behaves like $Ae^{\frac{-b+\sqrt{b^2-4mk}}{2m}t}$.

In the figure below, an example of case 1 and case 3 is shown together with the solution of case 2. We see, that case 2 is the one that decays fastest: it has the highest damping coefficient in its exponent. This is called critical damping. If you need to dampen unwanted oscillations: make sure you tune your damping parameter b such that $b^2 - 4mk = 0$.

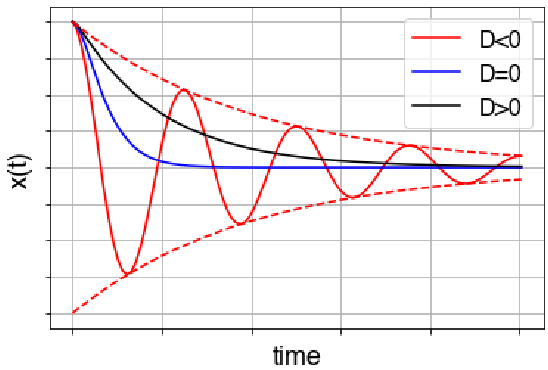


Figure 2.250: Different cases for the damped harmonic oscillator.

2.6.6.1 Evolution of the damping

Here we will have a quick look how the damping is evolving, that is we look at the root of the characteristic equation

$$\lambda_{1/2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \tag{2.375}$$

and see how it evolves as a function of the damping b in the complex plane.

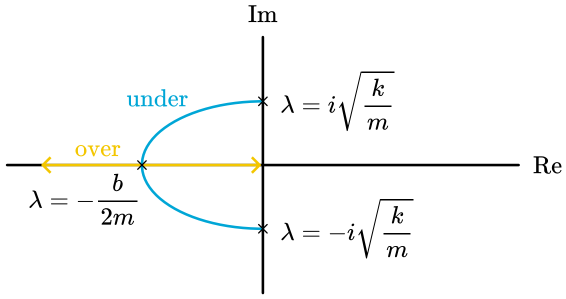


Figure 2.251: Evolution of λ as a function of b in the complex plane.

This gives quickly a qualitative view on the different regimes of the damping. The root $\lambda_{1/2}$ is in general complex. We start by looking at the value for roots $\lambda_{1/2}$ as a function of the damping b

- No damping: $b = 0$. The root is pure imaginary $\lambda_{1/2} = \pm i\sqrt{k/m}$ with two conjugate solutions on the imaginary axis. This gives pure oscillations.
- Some damping $0 < b < \sqrt{4mk}$. The root is complex, with real and imaginary part, the oscillation will damp out over time (shown in blue, underdamped regime).
- $b^2 = 4mk$. The roots collapse into one pure real root $\lambda = -b/2m$ (critically damped), no oscillation.
- Lots of damping $b > \sqrt{4mk}$. The root splits into two real roots, no oscillations (shown in yellow, overdamped regime).

The root walks over the shown graph from $b = 0$ on the imaginary axis to $b \to \infty$ over the blue and then yellow part of the graph. The yellow graph does not cross the imaginary axis.

From this plot you can directly see that the system is stable for $b > 0$, but unstable for $b = 0$ without the need to check the frequency that the system is driven with (for $b = 0$ driven with the resonance frequency results in an infinite amplitude - an instable system). How you can see that so quickly you will learn in the second year class *Systems and Signals*.

2.6.7 Driven Damped Harmonic Oscillator

Oscillators sometimes experience a driving force that can be periodic in itself. We will take here the case of a sinusoidal force with frequency ν. Once we understand this, forces consisting of more than one frequency (broader spectrum) can be understood using Fourier analysis (which you will learn about classes like *Systems and Signals* or *Fourier Analysis* in math). There you will also learn to treat this system in more detail analytically. Here we will stick to a simple driving force of the form $F_{ext} = F_0 \sin(\nu t)$.

This gives for the equation of motion:

$$m\ddot{x} + b\dot{x} + kx = F_0 \sin(\nu t) \tag{2.376}$$

with initial conditions: at $t = 0$ the particle will have some position x_0 and some velocity v_0.

The solution of the driven damped harmonic oscillator equation of motion for the case $D = b^2 - 4mk < 0$ is:

$$x(t) = Ae^{-\frac{b}{2m}t} \sin\left(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}t + \varepsilon\right) + x_{max} \sin(\nu t + \alpha) \tag{2.377}$$

With A and ε determined by the initial conditions.

The two other parameters x_{max} and α are fixed. We will give only the expression for x_{max}:

$$x_{max} = \frac{F_0}{\sqrt{\left(\omega_0^2 - \nu^2\right)^2 + \frac{b^2}{m^2}\nu^2}} \tag{2.378}$$

For $t \to \infty$, the second part, i.e., the term from the driving force $x_{max} \sin(\nu t + \alpha)$, survives as the exponential decay will have damped the first term. The oscillation will have frequency ν of the driving force. As can be seen, the amplitude of the motion is for longer times x_{max}.

If the driving frequency $\nu \sim \omega_0$, the amplitude increases strongly. Especially for small damping, i.e., small b, the amplitude will increase to high values. This phenomenon is called *resonance*:

$$ifb \to 0and\nu \to \omega_0 thenx_{max} \to \infty resonance \tag{2.379}$$

2.6.8 Coupled Oscillators

In this course we mostly only consider one oscillator, but of course there could be many that are coupled in one way or another. Already Christiaan Huygens considered them.

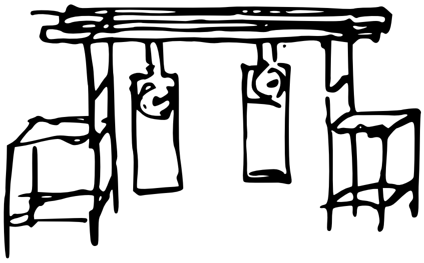


Figure 2.252: Huygens experiment of weakly coupled pendula.

There are 2 pendula suspended from a common connection, which rests on two chairs. If you set the pendula in motion, they will be initially *out of phase*, i.e. the relative position of the pendula is different. But over time their motion synchronises! What has happend? Apparently the two pendula are connected, *coupled*, via the suspension and act on each other, they are not independent, but influence the motion of the other pendulum.

The following video of weakly coupled metronomes below shows a modern day version of this phenomena.

Here the pendula are coupled via the ground. This influence is called *weak coupling*. In this course we cannot treat this coupling mathematically, but in the second year course on *Classical Mechanics* you will learn to study systems like these.

2.6.9 Examples

1. Example of resonance: sound waves are exciting a glass. By changing the frequency of the sound waves to the resonance frequency, the glass starts oscillating with increasing amplitude until it finally breaks.
2. Driven harmonic oscillator with damping.

Warning

App below needs updating

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation
from matplotlib.gridspec import GridSpec
from IPython.display import HTML

# Parameters
m = 1
k = 2
b = 0.5
F0 = 20
nu = 4
l0 = 200
```

```
dx = 0.28 * l0

# Derived parameters
w0 = np.sqrt(k / m)
w = np.sqrt(w0**2 - (b / (2 * m))**2)
phi = np.arctan(-nu * b / m / (nu**2 - w0**2))
xmax = F0 / m / np.sqrt((nu**2 - w0**2)**2 + (b * nu / m)**2)

A = dx - xmax * np.cos(phi)
B = (1 / w) * (-A * b / (2 * m) + xmax * nu * np.sin(phi))

# Time setup
t_stop = 20
fps = 30
dt = 1 / fps
t_vals = np.arange(0, t_stop, dt)

# Position function
def position(t):
    transient = A * np.exp(-b / (2 * m) * t) * np.cos(w * t) + B * np.exp(-b / 
(2 * m) * t) * np.sin(w * t)
    steady = xmax * np.cos(nu * t - phi)
    return l0 + transient + steady

x_vals = position(t_vals)

# Steady-state amplitude over range of driving frequencies
nu_range = np.linspace(0, 10, 400)
xmax_range = F0 / m / np.sqrt((nu_range**2 - w0**2)**2 + (b * nu_range / 
m)**2)

# Set up figure and gridspec
fig = plt.figure(figsize=(16, 4))
gs = GridSpec(1, 3, width_ratios=[1, 1.5, 1.5])
ax_anim = fig.add_subplot(gs[0])
ax_xt = fig.add_subplot(gs[1])
ax_amp = fig.add_subplot(gs[2])

# --- Animation axis setup ---
ax_anim.set_xlim(0, 500)
ax_anim.set_ylim(-50, 50)
mass_dot, = ax_anim.plot([], [], 'ro', markersize=15)
spring_line, = ax_anim.plot([], [], 'k-', lw=1.5)
# Horizontal L0 line from wall to equilibrium position
l0_line, = ax_anim.plot([0, l0], [0, 0], linestyle='--', color='gray', 
label='$L_0$')
ax_anim.legend(loc='upper right', fontsize=8, frameon=False)
time_text = ax_anim.text(0.02, 0.95, '', transform=ax_anim.transAxes)
ax_anim.set_xticks([])
ax_anim.set_yticks([])

# --- x(t) axis setup ---
xt_line, = ax_xt.plot([], [], 'b-')
ax_xt.set_xlim(0, t_stop)
ax_xt.set_ylim(np.min(x_vals) - 10, np.max(x_vals) + 10)
ax_xt.set_xlabel("$t$ (s)")
ax_xt.set_ylabel("$x$")
ax_xt.set_title("x(t) — Position over Time")
ax_xt.set_xticks([])
ax_xt.set_yticks([])

# --- Steady-state amplitude axis setup ---
amp_line, = ax_amp.plot(nu_range, xmax_range, 'g-')
nu_line = ax_amp.axvline(nu, color='red', linestyle='--', label='$\\nu$')
w0_line = ax_amp.axvline(w0, color='green', linestyle='--', label='$\
\omega_0$')
```

```python
ax_amp.set_xlim(0, 10)
ax_amp.set_ylim(0, np.max(xmax_range) * 1.1)
ax_amp.set_xlabel("Driving Frequency $\\nu$ (Hz)")
ax_amp.set_title("Steady-State Amplitude vs $\\nu$")
ax_amp.legend(loc='upper right', fontsize=8, frameon=False)
ax_amp.set_yticklabels([])
# Function to draw spring as a zig-zag between 0 and x
def get_spring_coords(x, num_zigs=12, amplitude=10, end_offset=7):
    x_end = x - end_offset  # shorten to avoid overshooting the mass
    xs = np.linspace(0, x_end, num_zigs + 1)
    ys = np.zeros_like(xs)
    ys[1:-1:2] = amplitude
    ys[2::2] = -amplitude
    return xs, ys

def init():
    mass_dot.set_data([], [])
    spring_line.set_data([], [])
    time_text.set_text('')
    xt_line.set_data([], [])
    return mass_dot, spring_line, time_text, xt_line, nu_line, w0_line

def animate(i):
    x = x_vals[i]
    mass_dot.set_data([x], [0])

    # Update spring line with zig-zag
    xs, ys = get_spring_coords(x)
    spring_line.set_data(xs, ys)

    time_text.set_text(f"$t$ = {t_vals[i]:.2f} s")
    xt_line.set_data(t_vals[:i], x_vals[:i])
    return mass_dot, spring_line, time_text, xt_line, nu_line, w0_line

ani = FuncAnimation(fig, animate, frames=len(t_vals), init_func=init,
blit=True, interval=1000*dt)

HTML(ani.to_jshtml())
```

3. 1940: the Tacoma Narrows Bridge in the state Washington on the West coast of the USA is brought into resonance by the wind. See the movie clip for the end result.
4. Breaking a HDD hard disk with a song of Janet Jackson

Read here about this truly amazing piece of applied physics on a blog of Microsoft developer Raimond Chen.

3. The blue sky: Rayleigh scattering (adapted from Mudde (2008)).

Light from the sun (and stars) will have to travel through the atmosphere before reaching the ground level. On its way it will be subject to absorption and scattering.
When you look on a clear day into the sky its color is blue, everybody knows that. But few people know why. The reason is found in the scattering properties of the molecules: the probability of light being scattered by an air molecule is proportional to the wave length of the light to the power −4, or rephrased: proportional to f^4 (f the frequency of the light, the theory of molecular scattering was given first given by Lord Rayleigh). Thus, blue light of a wavelength of 450nm is compared to red light (λ = 650nm) $(650/450)^4 = 4.4$ times more likely to be scattered. Consequently, the blue end from the (white) sun light has a reduced probability to reach our eye directly in comparison with the red end. And thus most of the scattered light that reaches us is blue: the sky is blue.
We will look at scattering of light by considering a simple molecule made of a fixed nucleus with one electron orbiting it. The equation of motion of the electron can be written as that of a harmonic oscillator, with eigen frequency ω_0:

$$m\ddot{x} + kx = 0 \rightarrow \ddot{x} + \omega_0^2 x = 0 \tag{2.380}$$

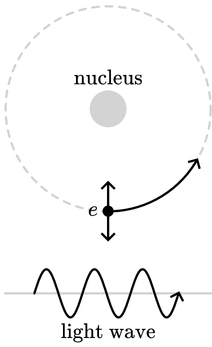


Figure 2.253: Simple model of electron-light scattering.

When light passes the electron, the electron feels a force since light is an electro-magnetic wave. The electric field is the dominating force. For light of wave length λ, i.e. angular frequency $\omega = 2\pi f = 2\pi \frac{c}{\lambda}$, the electric field can be written as $E_0 \sin \omega t$. Such a field will produce a force $F_e = eE_0 \sin \omega t$ on the electron, modifying its equation of motion to:

$$\ddot{x} + \omega_0^2 x = \frac{e}{m} E_0 \sin \omega t \tag{2.381}$$

We recognize this as the forced harmonic oscillator with solution

$$x(t) = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t + \frac{eE_0}{m} \frac{\sin \omega t}{\omega_0^2 - \omega^2} \tag{2.382}$$

The important part is the last one: the extra motion cause by the passing electric field. This causes an additional acceleration of the electron: $a(t) = -\frac{eE_0}{m} \frac{\omega^2}{\omega_0^2 - \omega^2} \sin \omega t$.
The electron in its original orbit does not radiate. However, due to the extra acceleration the electron starts radiating. It sends out an electromagnetic field with the wave length of the incoming light and an intensity proportional to the square of the acceleration, , i.e.

$$I \propto \left[\frac{\omega^2}{\omega_0^2 - \omega^2} \right]^2 \tag{2.383}$$

As the eigen frequency ω_0 of the electrons in oxygen and nitrogen is much higher than the frequency ω of the incoming light we have that this is basically proportional to $\left(\frac{\omega}{\omega_0}\right)^4$. As this radiation by the electron obviously feeds on the incoming light, we find that the scattering of the light is proportional to the frequency of the incoming light to the power 4.

6. Second-harmonic generation

Of course the harmonic potential is only a first order approximation around an equilibrium. An example, for a non-linear force or anharmonic potential effect, is the generation of second-harmonic generation. If you shine high intensity light onto the electrons of a molecule, they are pushed out of equilibrium further and if the governing potential is anharmonic, the electric field response will not only include the incoming frequency ω but also *higher harmonics* 2ω, 3ω, ..., but with much lower intensity. That the emitted frequencies are occurring in integer multiple of the incident frequency can be understood either from quantization of light into photons (and the conservation of energy) or from Fourier analysis of the periodic motion of the electron.

6. Erasmus Bridge & singing cables.

The bridge in Rotterdam, but also others, suffer from long cables that the wind can put into resonance. Their motion then generates acoustic waves in the audible spectrum. Listen here to the sound of the cables starting from 1:00 on the website for singing bridges!

2.6.10 Waves and oscillations

In the previous sections, we talked about oscillations of individual particles. Oscillations can also occur in a more collective mode. And there are plenty of examples: take for instance a violin or piano string. It is in essence an elastic string suspended between two fixed points. The string is under tension, that is: its natural length is (slightly) less than the distance between the two end points. As a consequence, equilibrium position of the string is a straight line and when brought out of equilibrium there is a net restoring force much like for the mass-spring system.

However, there are at least two important differences: (1) the restoring force is the net result from pulling on a small part of the string by its neighbor parts; (2) the entire string can oscillate in a direction perpendicular to the equilibrium position of the string, making the problem multi-dimensional.

We will give here an intuitive derivation of the equation of motion. Don't worry if you don't grasp it fully. This will come back in your studies further down the line.

In the figure below, a part of the string is drawn with special attention to a small part (the red line). On this small part the tension from the left and right side is pulling on the red part. This is visualized by the two blue arrows. In the inset, this is drawn at a larger scale. The two blue arrows are equal in magnitude (T) as the tension in the string is the same everywhere. But the direction in which the two blue forces are pulling is slightly different as the string is curved.

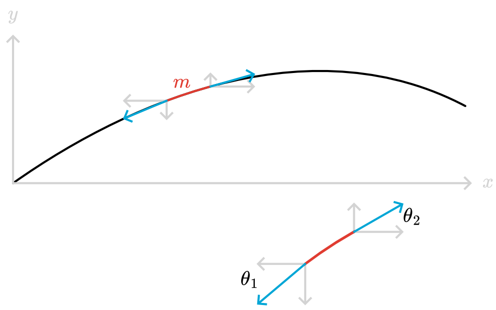


Figure 2.254: Forces on a small part of a string; inset shows an exaggeration of the vertical components of the forces.

If we call the angle of the blue forces with the x-axis θ, then $\theta_1 \neq \theta_2$. This makes that a net force is action on the small red piece. And according to Newton's second Law, the small red mass must accelerate.

Let's set up N2 for the red piece. The problem is 2-dimensional, so we set up N2 for the x and y-direction:

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -T\cos\theta_1 + T\cos\theta_2 \\ m\frac{d^2y}{dt^2} &= -T\sin\theta_1 + T\sin\theta_2 \end{aligned} \tag{2.384}$$

Next, we simplify by only looking at situations where the angle θ_1 and θ_2 are small. The we can approximate the sin and cos terms: if $\theta \ll 1$ then sin $theta \approx \theta$ and $\cos\theta \approx 1$ and we can write

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -T + T = 0 \\ m\frac{d^2y}{dt^2} &= -T\theta_1 + T\theta_2 \end{aligned} \tag{2.385}$$

Thus: for the x direction we don't need to worry, nothing interesting happening there.

For the y-direction we face that we have too many unknowns. We need relations between θ_1, θ_2, y and x. We are going to use again that $\theta \ll 1$ but know to make it seemingly more complex.

If $\theta \ll 1$ then $\tan\theta \approx \theta$. And we are going to replace θ by $\tan\theta$. Is that smart??? Now we get trigonometry back in the equation!! Don't worry. We use the $\tan\theta$ in another way. It is also the direction of the tangent to the curve the spring is making at the point where we are looking. In formula:

$$\tan\theta = \frac{dy}{dx} \tag{2.386}$$

And this is the coupling between angles and coordinates that we have been looking for.

We are going to plug this in in N2 for the y-direction. But before doing so: the left position of the red piece is at position x. So instead of label '1' we will use subscript x. Similarly, the right end of the red piece is at $x + dx$. Thus we can write

$$m\frac{d^2y}{dt^2} = -T\left[\frac{dy}{dx}\right]_x + T\left[\frac{dy}{dx}\right]_{x+dx} \tag{2.387}$$

It looks still pretty messy but we are almost there. The mass of the red piece obviously scales with its length. So if we introduce μ as the mass of the string per unit length, we can write for the mass of the red piece: $m = \mu dx$. Our equation can now be written as

$$\frac{d^2y}{dt^2} = \frac{T}{\mu}\frac{\left[\frac{dy}{dx}\right]_{x+dx} - \left[\frac{dy}{dx}\right]_x}{dx} \tag{2.388}$$

We recognize on the right hand side the second derivative of y with respect to x. Whereas on the left hand we see differentiating with respect to t.

$$\frac{d^2y}{dt^2} = \frac{T}{\mu}\frac{d^2y}{dx^2} \tag{2.389}$$

To make clear that we mean on the left hand side we mean: take the derivative only with respect to time we use ∂t instead of dt. Similarly on the right hand ∂x instead of dx. And we get our final result replacing $\frac{T}{\mu}$ by $math.sf(v)^2$

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2} \tag{2.390}$$

This equation is called the **wave equation** and you will find it back in many branches of science and engineering. To solve it, you need advance calculus and that will certainly come in future courses. Here we will look at some global aspects of the equation.

- units of $v^2 : m^2/s^2$. Thus v is a kind of velocity, at least based on its dimension.
- if $y(x,t)$ is such that it only depends on $x \pm vt$, that is $y(x,t) = y(x - vt)$ then no matter what y as function is, it is always a solution to the wave equation.

This is straightforward to prove: given $y(x,t) = y(x - vt)$ then call $s \equiv$

$$\frac{\partial y}{\partial t} = \frac{dy}{ds}\underbrace{\frac{\partial s}{\partial t}}_{= math.sf(-v)} \tag{2.391}$$

Note the meaning of ∂t: differentiate $s = x - math.sf(v)t$ as if x is a constant, not depending on t.

We can differentiate this once more:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(-math.sf(v)\frac{dy}{ds}\right) = -math.sf(v)\frac{d}{ds}\left(\frac{dy}{ds}\right)\frac{\partial s}{\partial t} = math.sf(v)^2\frac{d^2y}{ds^2} \tag{2.392}$$

Subsequently we look at $\frac{\partial^2 y}{\partial x^2}$:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{dy}{ds}\underbrace{\frac{\partial s}{\partial x}}_{=1}\right) = \frac{d}{ds}\left(\frac{dy}{ds}\right)\frac{\partial s}{\partial x} = \frac{d^2 y}{ds^2} \tag{2.393}$$

If we now substitute these two results in the wave equation we see:

$$\begin{gathered}\frac{\partial^2 y}{\partial t^2} - math.sf(v)^2\frac{\partial^2 y}{\partial x^2} = math.sf(v)^2\frac{d^2 y}{ds^2} - math.sf(v)^2\frac{d^2 y}{ds^2} = 0 \Rightarrow \\ \frac{\partial^2 y}{\partial t^2} = math.sf(v)^2\frac{\partial^2 y}{\partial x^2}\end{gathered} \tag{2.394}$$

And we see that our choice for $y(x,t) = y(x - math.sf(v)t)$ automatically obeys the wave equation.

From the above we also learn that if the string has a certain 'amplitude' y at position x on time t a little later this same amplitude will show up at a position a bit further along the string. Argument: given x and t then at (x,t) the amplitude of the string is $y(x - math.sf(v)t)$ and a little later, at $t + \Delta t$ we can look at position $x + math.sf(v)\Delta t$: there $y(x + \Delta x, t + \Delta t)$ is

$$y(x + \Delta x, t + \Delta t) = y(x + math.sf(v)\Delta t - math.sf(v)(t + \Delta t)) = y(x - math.sf(v)t) = y(x,t) \tag{2.395}$$

This actually means, that a traveling wave can be present in the string. We now this from our childhood when we probably all have been playing with a long rope making waves in it by quickly moving one end up and down.

The wave equation has as constant $math.sf(v)^2$. We have identified this as a velocity and we now understand that it is the velocity with which a wave travels. But since the equation contains the square of the velocity, we conclude that if we have a solution with $+math.sf(v)$, then also a solution with $-math.sf(v)$ holds. In other words: waves can travel in 2 directions and they do so with the same speed (in magnitude).

In the figure below, a wave is shown that starts as seemingly one hump. But it actually is two traveling waves on a rope.

Moreover, the rope has a fixed end at the left and a free one at the right. Notice the difference in reflection of the waves at both ends.

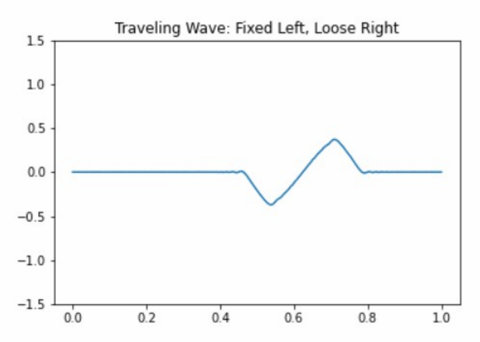


Figure 2.255: Forces on a small part of a string; inset shows an exaggeration of the vertical components of the forces.

2.6.10.1 Wave characteristics

Waves are omnipresent. We find them in musical instruments e.g. the violin but also in flutes where the wave is directly in the air in the instrument. We have them in water and air: waves on the oceans, waves when we speak. There are waves in solid materials for instance after an earthquake. We use waves in telecommunication.

Why are waves so generally found? They are the analogue of the harmonic oscillator. And thus, many systems in that are brought a bit out of equilibrium will try to go back to equilibrium, over shoot it and end up in a wavy motion.

Wave Length

Waves are often sinusoidal and if not, via Fourier Analysis they can be decomposed of a set of sinusoidal waves that built together the pattern we observe.

A sinusoidal wave is of the form

$$y(t) = A\sin(2\pi f t) \tag{2.396}$$

with f its frequency (and thus $\omega = 2\pi f$ its angular frequency).

As we have seen above, in general the wave is also a function of position:

$$y(x,t) \sim A\sin(x - vt) \tag{2.397}$$

How can we connect these two forms? First, we need to realize that the last equation has a dimensional issue: what is the sinus of say 7 meter? In other words, the argument of the sin-function should be dimensionless. So we write is in a different form, introducing the frequency in it:

$$y(x,t) = A\sin\left(\frac{2\pi f}{math.sf(v)}x - 2\pi f t\right) \tag{2.398}$$

This seems unnecessary complicated. But it is not! The factor $\frac{f}{math.sf(v)}$ has dimension 1 over length. If we call it $\frac{f}{math.sf(v)} \equiv \frac{1}{\lambda}$ we can write

$$y(x,t) = A\sin\left[2\pi\left(\frac{x}{\lambda} - f t\right)\right] \tag{2.399}$$

Interpretation: for a fixed value of t the wave is periodic in space with period λ. This is what we already know: the wave has a wave length λ.

On the other hand: for a fixed position x the point at x oscillates with a frequency f and thus has a period $T = \frac{1}{f}$. Note that λ and f are coupled to each other:

$$\lambda \cdot f = math.sf(v) \tag{2.400}$$

Example: Guitar string and frequency

Adapted from Pols (2021)

Guitar strings produce their sound by transverse resonant standing waves [2]. The natural frequency of a guitar string depends on the wavelength and the wave velocity: $f_0 = \frac{v}{\lambda_0}$ with $\lambda_0 = 2L$ where L is the length of the string. As shown above, the wave velocity is dependent on the string's tension and material: $v = \sqrt{\frac{F}{\mu}}$ where μ is the mass per unit length.

As every guitar player knows, changing the tension of the string, changes the frequency (tuning). One increases the tension by twisting the tuning knob which effectively stretches the string. The change in tension can be calculated using: $F = \sigma A = E\varepsilon A$, with $\varepsilon = \frac{\Delta l}{l_0}$, A the cross sectional area and E Young's modulus. Rearranging yields:

$$f^2 = \frac{EA}{4\mu L^3}\Delta l + f_0^2 \tag{2.401}$$

2.6.10.2 Standing waves versus traveling waves

If we look at the motion of the string on a violin closely, we will not see traveling waves running from one side of the string to the other. Instead, we see all parts of the string moving up and down collectively: they have formed a standing wave. that is a wave that does not travel, but has a fixed, stationary shape whose amplitude various with time.

For a string with two ends fixed like on a piano or violin, the string can only show standing waves that 'fit'. These standing waves are sinusoidal and their wave length should be such that the beginning and end of the string don't oscillate. In the figure below four possibilities are shown.

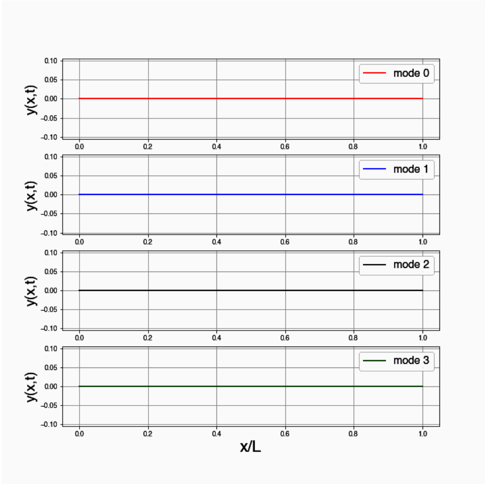


Figure 2.256: Standing waves in a string.

We see that there is a simple relation between the length of the string, L and the possible wave length, λ of the standing waves:

$$\frac{n}{2}\lambda = L \Leftrightarrow \lambda = \frac{2L}{n} with n \in N \tag{2.402}$$

Further we see that the smaller the wavelength, the faster the oscillation. This is due to the relation $\lambda \cdot f = math.sf(v)$ that still holds: $f = \frac{math.sf(v)}{\lambda} = \frac{n math.sf(v)}{2L}$.

The traveling waves had as mathematical form $\sin(x - math.sf(v)t)$. The standing waves take forms like $\sin \frac{x}{\lambda} \cdot \sin(2\pi f t)$. You will learn much more about this in e.g. Fourier Analysis classes.

2.6.10.3 Water waves and Sound waves

It is not necessary that a wave is caused by a tension in the material that tries to restore the equilibrium position. The restoring force can be of a different nature. A well know example is the water waves that we see on lakes and seas. Here gravity is the restoring force: it tries to pull a crest down and push a through up. The water inertia causes overshoot resulting in oscillations, that we call waves. In dealing with waves, we usually don't use the frequency f, but instead the angular velocity $\omega = 2\pi f$. Similarly, frequently the wave length λ is replaced by the wavenumber $k \equiv \frac{2\pi}{\lambda}$. Note that these two quantities are also related to each other by the speed of the waves: $\lambda \cdot f = \frac{2\pi}{\lambda}\frac{\omega}{2\pi} = \frac{\omega}{k} = math.sf(v)$.

For water waves (with large wave length) the angular momentum and the wave number are coupled to the depth, h, of the water:

$$\omega^2 = gk \tanh(kh) \tag{2.403}$$

From this we learn that waves on deep water travel much faster than on shallow water. This can be seen on our shores: the waves coming from the open sea are slowed down when the approach our beaches. But behind them the fast ones still come in. As a consequence, the wave gets squeezed in length and thus must get higher. This can be extreme with dramatic consequences: the Tsunami. The wave of the Tsunami is formed out in the open, where the sea is very deep. Here it travels at a very high speed which also means that it is a long wave. The Tsunami waves can travel at velocities of 200m/s and have wave length of hundreds of kilometers. However at full sea their amplitude is in the centimeter, decimeter range. A ship at full sea will hardly notice the passing Tsunami wave. But when the approach the shore, the front of the wave is slowed down to tens of m/s. As the back is still coming in at full speed the wave amplitude has to increase. And thus a huge wave in terms of amplitude storms towards the shore. A wall of water is seen coming, crushing everything in its way.

Sound waves are another type of waves that occur frequently. They can exist in solids, liquids and gasses. In contrast to the waves we have discussed so far, the amplitude is not perpendicular to the direction of traveling. It is what we call a longitudinal wave that oscillates in the same direction as it moves. The other waves are called transversal waves.

For sound waves it is the pressure that is the restoring force. The ‘crest’ is compressed material, the ‘through’ is an expansion part. Newton was intrigued by sound waves and provided a theory for them. He found that the speed of sound in air, according to his theory, was about 290 m/s. In reality it is some 340 m/s. Newton was well aware of the mismatch. But he couldn’t find a good explanation. It took another 100 years for Pierre Laplace corrected Newton’s work and arrived at the correct answer. Newton did not know that sound is connected to adiabatic compression. He couldn’t as the entire concept was not know. Laplace realized that Newton basically had made an isothermal solution and corrected this.

2.6.11 Examples, exercises & solutions

Updated: 04 feb 2026

2.6.11.1 Worked examples

2.6.11.1.1 Title of example

Interpret the problem

HIER DE INTERPRETATIE

Develop the solution

HIER DE DEVELOPMENT

Evaluate the problem

HIER DE EVALUATE

Assess the problem

HIER DE ASSESS

2.6.11.2 Exercises

Here are some exercises that deals with oscillations. Make sure you practice IDEA.

2.6.11.3 Answers

2.6.11.4 Experiment & Simulation

Mass spring

Find a rubber band and use nothing but a mass (that you are not allowed to weigh) that you can tie one way or the other to the spring, a ruler, and the stopwatch/clock on your

Exercise 2.257: 🌶️

A massless spring (spring constant k) is suspended from the ceiling. The spring has an unstretched length l_0. At the other end is a point particle (mass m).

- Make a sketch of the situation and define your coordinate system.
- Find the equilibrium position of the mass m.
- Set up the equation of motion for m.
- Solve it for the initial condition that at $t = 0$ the mass m is at the equilibrium position and has a velocity v_0.

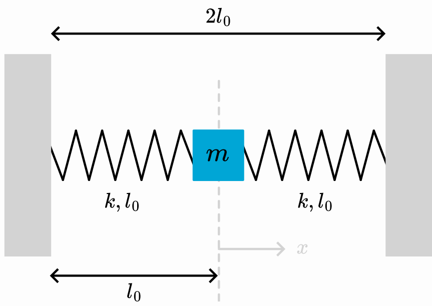
Exercise 2.258: 🌶️🌶️

Same question, but now two springs are used. Spring 1 has spring constant k; spring 2 has $2k$. Both have the same unstretched length l_0.

- The two springs are used in parallel, i.e., both are connected to the ceiling, and m is at the joint other end of the springs.
- Both springs are in series, i.e., spring 1 is suspended from the ceiling, and the other one is attached to the free. The particle is fixed to the free end of the second spring.

Exercise 2.259: 🌶️

A mass m is attached to two springs. The other ends of the springs are fixed and cannot move. The distance between these point is $2l_0$. The mass can move only in the horizontal direction and there is no gravity. See the figure below for a sketch.



The springs are identical: both have rest length l_0 and spring constant k. Based on symmetry, we take the origin in the center of the figure.

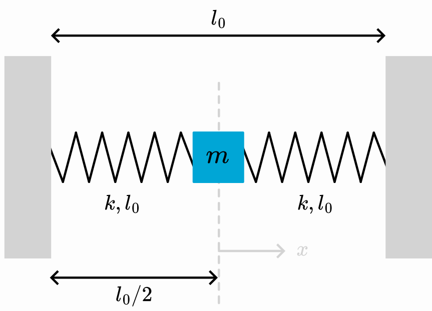
We are going to repeat the same analysis as in the previous exercises.

- Make a sketch of the situation and define your coordinate system.
- Find the equilibrium position of the mass m.
- Set up the equation of motion for m.
- Solve it for the initial condition that at $t = 0$ the mass m is at the equilibrium position and has a velocity v_0.

mobile.Set up an experiment to find the mass , the spring constant , and the damping coefficient .Don't forget to make a physics analysis first, a plan of how to find both and .From Wikimedia Commons: bands, CC-SA 4.0; apple, CC-BY 2.0, ; phone, PD; ruler, CC-BY 4.0.

Exercise 2.261: 🌶️ 🌶️

The same as above, but now the length between the two point where the spring are attached to is l_0 instead of $2l_0$.



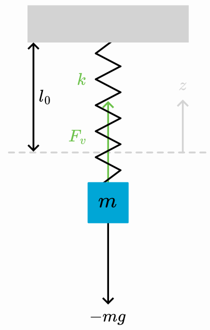
Note

In the figure k, l_0 denotes the characteristics of the springs.

- Make a sketch of the situation and define your coordinate system.
- Find the equilibrium position of the mass m.
- Set up the equation of motion for m.
- Solve it for the initial condition that at $t = 0$ the mass m is at the equilibrium position and has a velocity v_0.

Solution 2.263: Solution to Exercise 1

Sketch; $z = 0$ is when the mass is l_0 below the ceiling.



Equilibrium position of the mass m:

$$\sum F = 0 \to F_v - mg = 0 \tag{2.404}$$

Force of the spring: $F_v = -k(l - l_0) = -kz$. Thus

$$-kz_{eq} - mg = 0 \to z_{eq} = -\frac{mg}{k} \tag{2.405}$$

Equation of motion for m: set up N2

$$m\frac{dv}{dt} = -kz - mg \tag{2.406}$$

Solution with $z(0) = z_{eq}$ and $v(0) = v_0$:

homogeneous part of the equation: $m\frac{dv}{dt} + kz = 0$

$$z_{hom}(t) = A\cos\omega_0 t + B\sin\omega_0 t \tag{2.407}$$

with $\omega_0^2 = \frac{k}{m}$

special solution: $z_s = -\frac{mg}{k}$

general solution:

$$z(t) = z_{hom}(t) + z_s(t) = z_{hom}(t) = A\cos\omega_0 t + B\sin\omega_0 t - \frac{mg}{k} \tag{2.408}$$

initial conditions:

$$z(0) = z_{eq} = -\frac{mg}{k} \to A = 0 \tag{2.409}$$

and

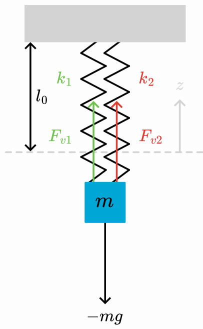
$$v(0) = v_0 \to v_0 = \omega_0 B \to B = \frac{v_0}{\omega_0} \tag{2.410}$$

Thus, the solution is

$$z(t) = -\frac{mg}{k} + \frac{v_0}{\omega_0}\sin\omega_o t \tag{2.411}$$

Solution 2.265: Solution to Exercise 2

Sketch; $z = 0$ is when the mass is at l_0 below the ceiling. Now we have 2 springs, one with spring constant k_1, the other with k_2. Both have the same rest length l_0



Equilibrium position of the mass m:

$$\sum F = 0 \rightarrow F_{v1} + F_{v2} - mg = 0 \tag{2.412}$$

Forces of the springs: $F_{v1} = -k_1(l - l_0) = -k_1 z$ and $F_{v2} = -k_2(l - l_0) = -k_2 z$. Thus

$$-k_1 z_{eq} - k_2 z_{eq} - mg = 0 \rightarrow z_{eq} = -\frac{mg}{k_1 + k_2} \tag{2.413}$$

Equation of motion for m: set up N2

$$m\frac{dv}{dt} = -(k_1 + k_2)z - mg \tag{2.414}$$

Thus we conclude, that the exercise is basically the same: all we have to do is replace k by $K_{tot} = k_1 + k_2$

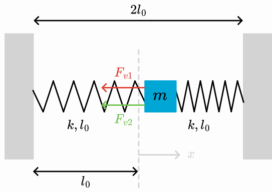
$$m\frac{dv}{dt} = -k_{tot}z - mg \tag{2.415}$$

The solution with $z(0) = z_{eq}$ and $v(0) = v_0$ is thus

$$z(t) = -\frac{mg}{k_{tot}} + \frac{v_0}{\omega_0}\sin \omega_o t \tag{2.416}$$

with $\omega_0^2 = \frac{k_{tot}}{m}$

Solution 2.267: Solution to Exercise 3



Again, we have two springs acting on the mass. However, they are no on opposite sides. We expect on symmetry arguments that the equilibrium will be in the middle, i.e at $x = 0$.

If the mass is positioned to the right of $x = 0$, spring 1 is extended beyond its rest length and will pull in the negative x-direction:

$$F_{v1} = -k(l - l_0) = -kx \tag{2.417}$$

Spring 2 will than be shorter than its rest length and will push to the negative x-direction:

$$F_{v2} = k(l - L_0) = -kx \tag{2.418}$$

Thus, equilibrium is reached when

$$\sum F = F_{v1} + F_{v2} = 0 \to -2kx = 0 \to x_{eq} = 0 \tag{2.419}$$

as we anticipated.

Equation of motion for m: set up N2

$$m\frac{dv}{dt} = -kx - kx = -2kx \tag{2.420}$$

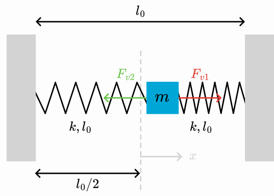
Thus we conclude, that the exercise is basically the same: all we have to do is replace k by $k_{tot} = 2k$

$$m\frac{dv}{dt} = -2kx \tag{2.421}$$

General solution $x(t) = A\sin\omega_0 t + B\cos\omega_0 t$ with $\omega_0^2 = \frac{2k}{m}$.

Like in the previous exercises, it is now a matter of specifying the initial conditions and finding A and B.

Solution 2.269: Solution to Exercise 4



Again, we have two springs acting on the mass. Now they don't fit both with their rest length. The will be compressed and try to lengthen. However, based on symmetry we still do expect that $x = 0$ is the equilibrium position.

If the mass is positioned to the right of $x = 0$, spring 1 still too short and will push to the right:

$$F_{v1} = -k(l - l_0) = -k\left(\frac{l_0}{2} + x - l_0\right) = k\left(\frac{l_0}{2} - x\right) \tag{2.422}$$

Spring 2 will than be even shorter and will push to the negative x-direction:

$$F_{v2} = k\left(\frac{l_0}{2} - x - l_0\right) = -k\left(\frac{l_0}{2} + x\right) \tag{2.423}$$

Thus, equilibrium is reached when

$$\sum F = F_{v1} + F_{v2} = 0 \to k\left(\frac{l_0}{2} - x\right) - k\left(\frac{l_0}{2} + x\right) = -2kx = 0 \to x_{eq} = 0 \tag{2.424}$$

as we anticipated.

Equation of motion for m: set up N2

$$m\frac{dv}{dt} = -kx - kx = -2kx \tag{2.425}$$

Thus we conclude, $k_{tot} = 2k$, which is identical to the previous exercise!

Exercise 2.271: Simulation

If the force acting on a particle is conservative, a potential energy can be defined. The sum of kinetic and potential energy is then constant: (mechanical) energy is conserved. However, if the force is not a conservative one, mechanical energy is not conserved. The force will perform work on the particle. The total energy, if we include the work done, is still conserved. In many real life cases, the work done by non-conservative forces shows up as heat.

Energy conservation of a mass-spring
First, we will consider a mass m, suspended on a spring (rest length l_0 en spring constant k). The spring is at a fixed position at one end, while the mass m is attached at the other end. m is displaced by an amount Δx and then released without initial velocity.

Task:
Set up the analysis of this problem. It can be solved analytically.

- Make a sketch
- Set up a model
- Solve the model
- Show that energy is conserved: the sum of kinetic and potential energy is constant.
- Make a python programme that outputs a plot of the kinetic, potential and total energy as a function of time.

Friction: work done and energy conservation We could introduce a frictional force that will result in damping of the oscillation. A common example is having a friction that is proportional to the velocity of the mass m and works against the direction of motion:

$F_f = -bv$

with b a proportionality constant. Note that we are taking the problem as one-dimensional.

We could solve this problem analytically, but it is illustrative to do it numerically as we can then easily compute the work done the friction force.

So, the **task** is:

- Make a sketch
- Set up a model
- Make a python programme that outputs a plot of the kinetic, potential and total mechanical energy (that is the sum of kinetic and potential energy) as a function of time.
- Compute the work done by the friction force and plot this also as function of time.

Take the following parameters: $l_0 = 20$ cm, $\Delta x = 1$ cm, $m = 1$ kg, $k = 1$ N/m, $b = 0.1$ Ns/m

It is instructive to change the time step dt while keeping the total time (that is $N \cdot dt$) constant. You will notice that for 'large' time steps, it seems that energy is not conserved (of course taking into account the work done by the friction force), but that by making the time step smaller conservation of energy seems to be more and more obeyed. This is a consequence of *numerical errors*. The laws of physics are clear: energy must be conserved.

2.7 Collisions

Updated: 04 feb 2026

2.7.1 What are collisions?

In daily life we do understand what a collision is: the bumping of two objects into each other. From a physics point of view, we see it slightly different. The objects don't have to touch. It is sufficient if they undergo a mutual interaction *'with a beginning and an end'*. What do we mean by this?

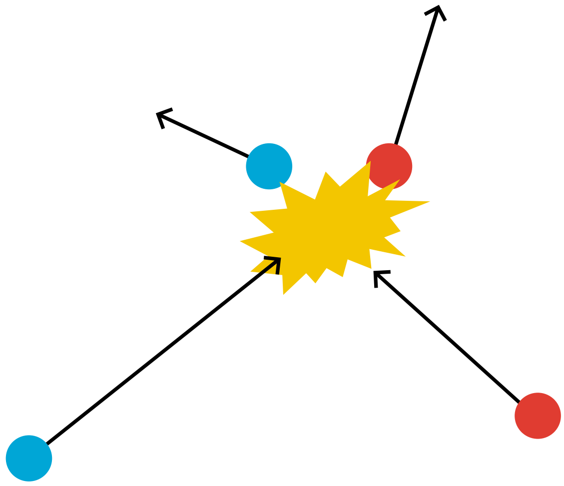


Figure 2.272: Collision of two particles.

Firstly, the mutual interaction means that the objects interact with each other through a mutual force, i.e. a force (pair) that obeys Newton's third law.

Secondly, we assume that this force works over a small distance only. Or re-phrased we will only consider the situation before the objects feel the force and compare that to after they have felt it. We don't bother about the details of the motion of the objects *during* their interaction. Hence, when we depict a collision as in Figure 1, we usually draw the situation before the collision, then some kind of 'comic way' of showing the collision and finally we draw the outcome of the collision, so after the interaction. In many cases, people leave the middle part out of their drawing.

There are two principle types of collisions to distinguish: *elastic* and *inelastic* collisions. For an elastic collision the kinetic energy is conserved, whereas for an inelastic that is not the case. In the latter case, energy can be converted into deformation or heat.

Since the objects interact under the influence of their mutual interaction, we have conservation of momentum:

$$\sum_i \vec{p}_i^{\,before} = \sum_i \vec{p}_i^{\,after} \tag{2.426}$$

Why? No external forces implies constant total momentum.

Shooting coins

Line up two coins on a table, placed edge to edge. Then position a third coin in front of them, a short distance away. Push the coin in the middle hard down.Now use your other hand to flick the third coin strongly, aiming to hit the middle coin and cause a collision.Try varying the setup—for example, using a heavier coin or a lighter one as the target—and observe the effects.What happens? Compare the velocities, qualitatively, before and after the collision.

2.7.2 Elastic Collisions

For an elastic collision the kinetic energy is conserved by definition (next to the conservation of momentum). That is the sum of the kinetic energy before the collision is the same as the sum after the collision. This type of collision is also called *hard-ball collision*: as with colliding billiard balls no energy is dissipated into heat or deformation.

Figure 2.273: A simulation on collisions. Try to change the mass, velocity, angle of contact...

For elastic collisions the interaction force needs to be conservative. Then, a potential energy exists. And this energy is such that the objects have the same potential energy before as after the collision. Consequently energy conservation leads to:

$$E_{kin,before} + V_{before} = E_{kin,after} + \underbrace{V_{after}}_{=V_{before}} \Rightarrow E_{kin,before} = E_{kin,after} \quad (2.427)$$

2.7.2.1 Solving collision problems

Given a collision experiment where the initial situation before the collision is known, how do we compute the situation after the collision? What will the velocities of the object be?

Consider a head-on collision of two point particles on a line as shown in Figure 4. One particle with mass $3m$ is initially at rest ($u = 0$), the other with mass $2m$ has velocity $2v$. What are the velocities v', u' of the particles after the collision?

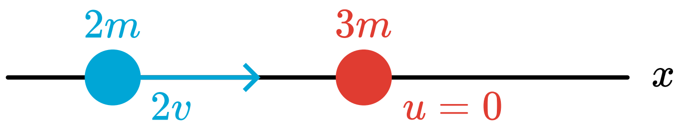


Figure 2.274: Example of a 1D elastic collision.

We can write down two equations using conservation of momentum and kinetic energy before and after the collision

$$\begin{array}{rcl} 2m(2v) + 0 & = & 2mv' + 3mu' \quad (*) \\ \frac{1}{2}2m(2v)^2 + 0 & = & \frac{1}{2}2mv'^2 + \frac{1}{2}3mu'^2 \end{array} \quad (2.428)$$

We have two equations and two unknowns (v', u'), therefore we can in principle solve this problem. The question is, what is the best strategy to do so? A strategy is needed especially since one equation involves the square of the velocity.

We first bring the velocities v, v' and u, u' to the same side.

$$\begin{array}{rcl} 2m(2v - v') & = & 3mu' \\ \frac{1}{2}2m(4v^2 - v'^2) & = & \frac{1}{2}3mu'^2 \end{array} \quad (2.429)$$

Now we divide the two equations (verify yourselves!), this makes the solution very easy as the squares of the velocities always divide out.

$$\Rightarrow 2v + v' = u' \quad (**) \quad (2.430)$$

We can use this to find both unknowns by smartly adding equations $(*)$ and $(**)$. Smartly in the sense that we can multiply either of the equations with a scalar in such way that one quantity can be discarded.

$$\begin{array}{ll} 4v & = 2v' + 3u' \\ 2v & = -v' + u' \mid *2 \\ ---8v & = 5u' \\ & \Rightarrow u' = \frac{8}{5}v \qquad 4v \ = 2v' + 3u' \\ 2v & = -v' + u' \mid * -3 \\ ----2v & = 5v' \\ & \Rightarrow v' = -\frac{2}{5}v \end{array} \quad (2.431)$$

This solution strategy always works. NB: you need to practice this. Although it is conceptually easy, we often see that students have problems when actually solving for the 2 unknowns.

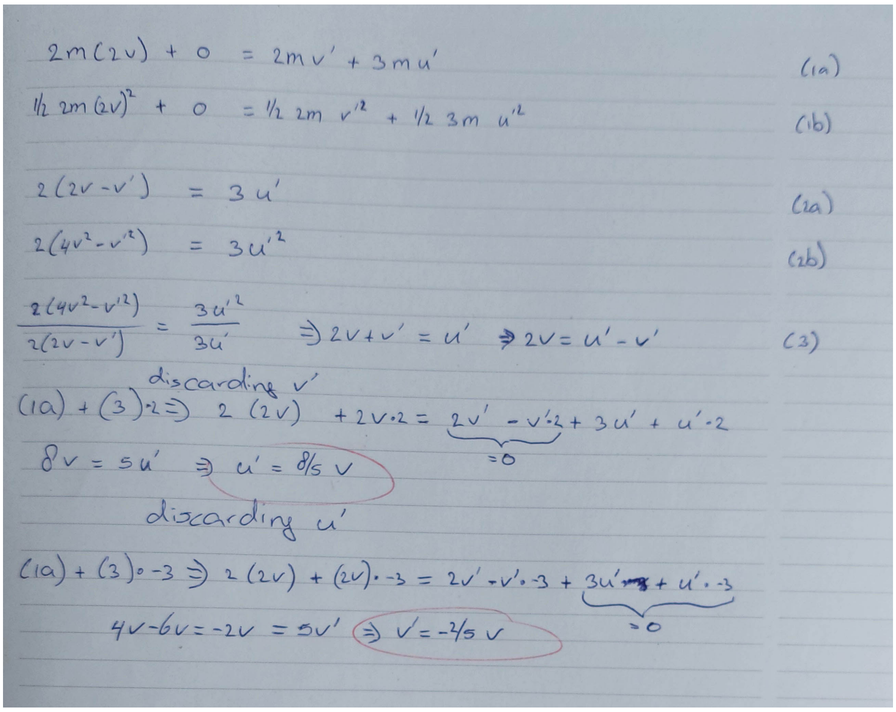


Figure 2.275: Solving the problem

Vpython simulation

Above we provided a Vpython simulation. Change the code in order to verify the above solution.

Actually, now we think about this strategy: isn't it strange, the kinetic energy equation is squared in our unknowns. Shouldn't we expect 2 solutions?

The answer is yes: there ought to be 2 solutions. Where is the second one? Note that when dividing the two equations, we have to make sure that we do not divide by 0. It is very well possible that we do so: suppose $u' = 0$, then also $2v - v' = 0$ and we cannot do the division. But what does that mean: $u' = 0$ and $2v - v' = 0$? Well, of course, then we have after the collision that the incoming particle $2m$ still has velocity $2v$ and the other particle $3m$ is still at rest.

In retrospect: of course this must be one of the solutions to the problem. We haven't specified anything about the interaction force. But suppose it is absent, that is, the particles don't interact at all. Then of course the situation before the collision (a bit strange calling this a collision, but anyway), will still be present after the 'collision'. If nothing happens to the particles, then obviously the sum of the momentum as well as of the kinetic energy stays the same. This actually provides a great check of your work: do you recover the initial conditions?

2.7.2.2 Collisions in a plane

Consider now a 2D elastic collision such that the two particles collide in the origin, Figure 6.

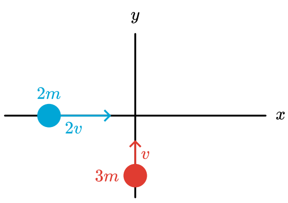


Figure 2.276: Example of a 2D elastic collision.

If we write down the equation of conservation of momentum (in x, y components) and of kinetic energy, we get

$$
\begin{array}{rcl}
2m(2v)+0 & = & 2mv'_x + 3mu'_x \\
0+3mv & = & 2mv'_y + 3mu'_y \\
\frac{1}{2}2m(2v)^2 + \frac{1}{2}3mv^2 & = & \frac{1}{2}2mv'^2 + \frac{1}{2}3mu'^2
\end{array}
\tag{2.432}
$$

Now we have **4** unknowns (v'_x, v'_y, u'_x, u'_y) but only **3** equations. The outcome seems not to be determined by the initial condition... Of course, that cannot be the case (Think shortly why?). The outcome is fully determined by the initial conditions, but we did not set up the equations in the best way because we did not first transform the problem into a 1D problem such that the collision is head-on.

We can use a Galilean Transformation to put one particle at rest. Here we set the blue particle to rest by subtracting $-2v$ from its velocity, that is we move with the blue particle (prior to the collision). The corresponding Galilean Transformation is

$$
\begin{array}{rcl}
x' & = & x - 2vt \\
y' & = & y
\end{array}
\tag{2.433}
$$

The red particle now has velocity $(-2v, v)$. The problem is still 2D.

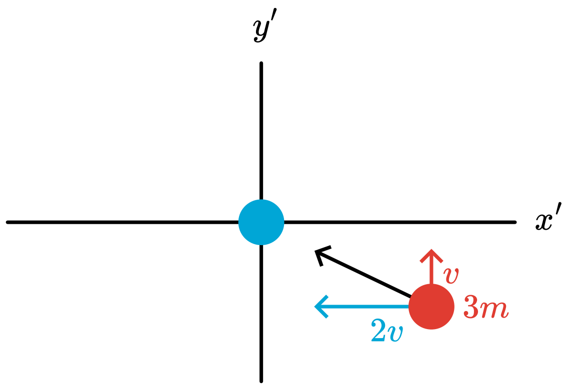


Figure 2.277: Applying the Galilean Transformation.

Next, we can rotate the coordinate system, to obtain a 1D head-on collision that we can solve as above.

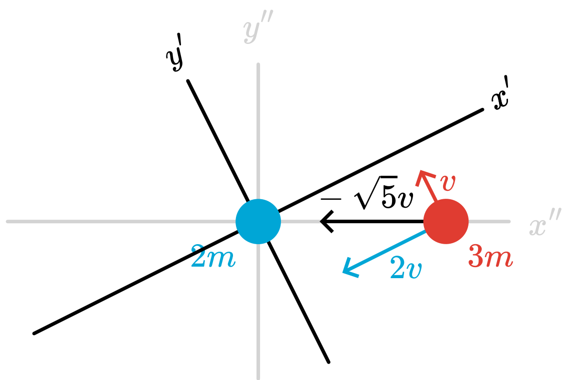


Figure 2.278: Rotating the coordinate system.

We see that we now have a 1-dimensional elastically collision with a particle of mass $3m$ coming in over the x"-axis with velocity $-\sqrt{5}v$. It will collide with a particle of mass $2m$ which is at rest. We know how to solve this problem and find the velocities of both particles after the collision. If we do this, we find that after the collision the velocity of the blue particle is $U''_{x''} = -\frac{6}{5}\sqrt{5}v$ and of the red particle $V''_{x''} = -\frac{1}{5}\sqrt{5}v$. Note that we have specified these velocity in the $(x", y")$ coordinate system.

Next steps would be to convert the velocities back to the initial coordinate frame. That is a bit cumbersome, but again conceptually easy. The final answer in the original frame of reference is:

$$
\begin{aligned}
2m: \ v'_x &= -\tfrac{2}{5}v, \ \ v'_y = \tfrac{6}{5}v \\
3m: \ \ u'_x &= \tfrac{8}{5}v, \ \ \ u'_y = \tfrac{1}{5}v
\end{aligned}
\tag{2.434}
$$

Figure 2.279: The 3Blue1Brown series on linear algebra describes the linear transformations above in a mathematical way. Using linear algebra, above computations will become easier.

2.7.3 Collisions in the Center of Mass frame

There is a special frame of reference: the Center of Mass (CM) frame. Its origin is defined with respect to the *lab frame* as

$$
\vec{R} = \frac{\sum m_i \vec{x}_i}{\sum m_i} \tag{2.435}
$$

We will introduce this formally in the next chapter.

As the mass is conserved during a collision we have

1. $\sum m_i = const$ and
2. as the momentum is conserved $\sum m_i \dot{\vec{x}}_i = const$,

we see that the velocity of the CM does not change before and after collision

$$
\dot{\vec{R}}_{before} = \dot{\vec{R}}_{after} \tag{2.436}
$$

Instead of working in the lab frame we can use the CM frame. A sketch of the coordinates and vectors is given in the figure below.

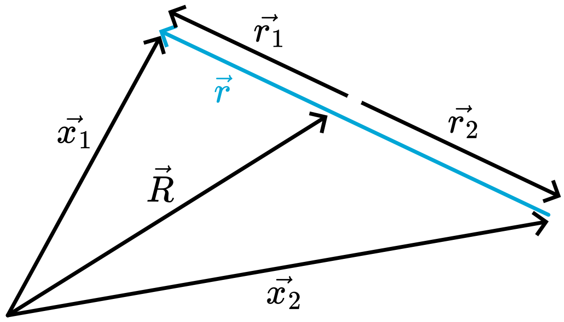


Figure 2.280: Center of mass.

For the relative coordinates $\vec{r}_i$ it holds that $\sum m_i \vec{r}_i = 0$. Considering two particles: The relative distance $\vec{r} = \vec{r}_1 - \vec{r}_2 = \vec{x}_1 - \vec{x}_2$ is Galilei invariant.

Using this property we express the vectors $\vec{r}_1$ and $\vec{r}_2$ in terms of the relative distance vector $\vec{r}$ for $\vec{r}_1 = \frac{\mu}{m_1}\vec{r}$ and $\vec{r}_2 = -\frac{\mu}{m_2}\vec{r}$ with μ the so-called reduced mass (see next chapter). Therefore

$$
\begin{aligned}
m_1 \vec{r}_1 &= \mu \dot{\vec{r}}_1 \\
m_2 \vec{r}_2 &= -\mu \dot{\vec{r}}_2
\end{aligned}
\tag{2.437}
$$

This means the momenta of both particles are *always* equal in magnitude and opposed in direction in the CM frame. Only the orientation of the pair $\dot{\vec{r}}_{1,2}$ can change from before to after the collision.

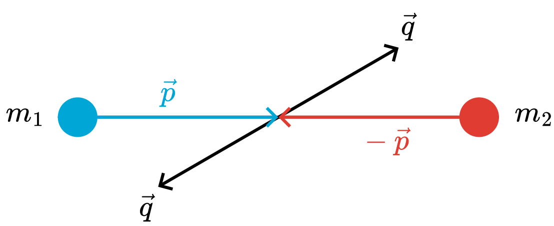


Figure 2.281: Collision in center of mass frame.

Exercise 2.282:

Add to the vpython code the center of mass and show that the velocity of the center of mass does not change.

2.7.4 Computational

For collisions with only a 'few' particles, it is doable to calculate the outcomes by hand. That is, if there are no angles involved. It becomes more difficult if we want, for instance, compute a box with 10^4 particles. Such a simulation may provide key insights in thermodynamic behavior.

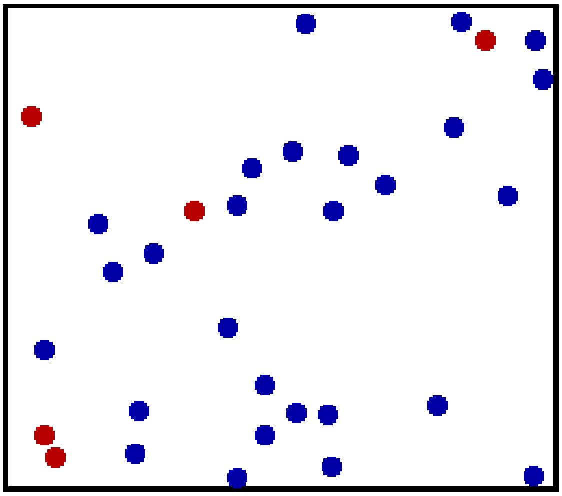


Figure 2.283: Computing multiple collisions by hand is quite challenging, but can be done 'easily' with the computer. Figure made by A. Greg, public domain

In computing collisions, we can make use of the general solution:

$$
\begin{aligned}
v'_{1x} &= \frac{v_1\cos(\theta_1-\phi)(m_1-m_2)+2m_2v_2\cos(\theta_2-\phi)}{m_1+m_2}\cos(\phi)+v_1\sin(\theta_1-\phi)\cos(\phi+frac(\pi,2)) \\
v'_{1y} &= \frac{v_1\cos(\theta_1-\phi)(m_1-m_2)+2m_2v_2\cos(\theta_2-\phi)}{m_1+m_2}\sin(\phi)+v_1\sin(\theta_1-\phi)\sin(\phi+frac(\pi,2)),
\end{aligned}
\tag{2.438}
$$

as derived in Craver (2013).

Figure 2.284: Figure made by Simon Steinmann, CC-SA

In an angle-free representation - using vectors rather than angles, the changed velocities are computed using the centers $x1$ and $x2$ at the time of contact as:

$$\begin{aligned} \boldsymbol{v}'_1 &= \boldsymbol{v}_1 - \frac{2m_2}{m_1+m_2}\ \frac{\langle \boldsymbol{v}_1-\boldsymbol{v}_2, \boldsymbol{x}_1-\boldsymbol{x}_2\rangle}{\mid \boldsymbol{x}_1-\boldsymbol{x}_2\mid^2}\ (\boldsymbol{x}_1-\boldsymbol{x}_2), \\ \boldsymbol{v}'_2 &= \boldsymbol{v}_2 - \frac{2m_1}{m_1+m_2}\ \frac{\langle \boldsymbol{v}_2-\boldsymbol{v}_1, \boldsymbol{x}_2-\boldsymbol{x}_1\rangle}{\mid \boldsymbol{x}_2-\boldsymbol{x}_1\mid^2}\ (\boldsymbol{x}_2-\boldsymbol{x}_1) \end{aligned} \tag{2.439}$$

In Python this would become:

```
vA_new = vA - 2 * mB / (mA + mB) * np.dot(vA - vB, rA - rB) /
(1e-12+np.linalg.norm(rA - rB))**2 * (rA - rB)
vB_new = vB - 2 * mA / (mA + mB) * np.dot(vB - vA, rB - rA) /
(1e-12+np.linalg.norm(rB - rA))**2 * (rB - rA)
```

Note that a vary small number is added ($1e-12$) to prevent that the denominator does not become 0.

2.7.5 Inelastic Collisions

For inelastic collisions only the *momentum is conserved*, but *not the kinetic energy*. The kinetic energy after the collision is always less than before the collision. As the total energy is conserved, some kinetic energy is converted to deformation or heat.

The amount of “inelasticity” or loss of energy can be quantified by the *coefficient of restitution e*

$$e \equiv -\frac{v_{rel,after}}{v_{rel,before}} \tag{2.440}$$

$$e^2 \equiv \frac{E_{kin,after}}{E_{kin,before}} in CM frame \tag{2.441}$$

For $e = 0$ the collision is fully inelastic, for $e = 1$ it is fully elastic.

Heat

In chapter Work & Engergy we have seen that energy is a conserved quantity. In inelastic collisions the kinetic energy is not conserved, that is, with every collision the temperature of both objects will increase. Remember from secondary school that heat can be calculated using

2.7.6 Exercises, examples & solutions

Updated: 04 feb 2026

2.7.6.1 Worked examples

2.7.6.1.1 Title of example

Interpret the problem

HIER DE INTERPRETATIE

Develop the solution

HIER DE DEVELOPMENT

Evaluate the problem

HIER DE EVALUATE

Assess the problem

HIER DE ASSESS

2.7.6.1.2 Newton's Cradle

Click on the link for an applet on Newton's cradle (gives you also the possibility to 'play' with different numerical solvers, from (too) simple to advanced).

2.7.6.2 Exercises

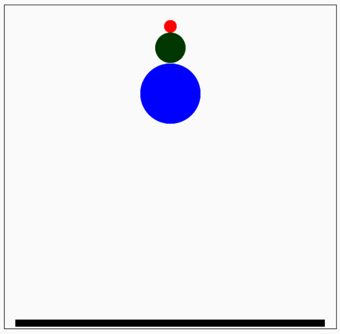
2.7.6.3 Experiment

restitution coefficient

Is the restitution coefficient of a bouncing tennis ball a constant or does it depend on the velocity at bouncing?
You can 'easily' find out yourself. What you need is a tennis ball, and your mobile with the phyphox app.Experiment: drop a tennis ball with zero initial velocity from various height, . Use the acoustic chronometer to measure the time between multiple bounces.Show that the relation between height and time between two bounces is equal to Use your recordings to compute the height as function of number of bounces and compute the restitution coefficient .Plot as a function and you will have answered the above question.

Exercise 2.285: Colliding Superballs 🌶️

Watch this video on bouncing superballs. We discussed this problem in this chapter.



Do you agree with the explanation in the movie?

We seem to violate the conservation of kinetic energy: there is much more kinetic energy after the collision than before! Can you figure out what is happening?

Tip

Look carefully at the bouncing of the blue ball with the earth. Is it really true that the velocity after bouncing is v and that the earth does not move? Probably not, as this violates conservation of momentum!

Exercise 2.287: 1D elastic collision 🌶️

Consider two particles, m_1 and m_2, moving along the x-axis. The two particles will elastically collide. We set mass 1 at a value of 1 (kg) and set m_1 to 6 (kg).

Solve the collision by using conservation of momentum and kinetic energy and compare your results with those of the widget.

You can change the value of m_1 and of the velocities of both particles before the collision. Change the values, predict what will happen, and check your prediction.

Exercise 2.288: 2D elastic collision 🌶️

Next, we consider an elastic collision between m_1 and m_2, but now in a 2-dimensional setting.

In the widget below, the situation is animated. You can change the values of the initial velocity and masses. Can you (qualitatively) predict the outcome of the collision for a given set os parameters?

Exercise 2.289: Inelastic Collision 🌶️

Particle m_1 is moving over the x-axis with unit velocity. Simultaneously, particle m_2 is moving over the y-axis also with unit velocity. Both particles will collide in the origin. The collision is inelastic with restitution coefficient e.

The widget below shows the trajectories of the particles and gives the velocities after the collision. Moreover, als the angle of the trajectories after the collision with the x-axis is given.

Can you solve this problem for a few values of the restitution coefficient? The 'easy ones' are for $e = 0$.

Exercise 2.290: Completely inelastic collision

Consider a particle with mass M being at rest in your frame of reference. A second particle of mass m comes in over the negative x-direction with velocity v. The collision is completely inelastic.

Find the velocities after the collision.

Exercise 2.291: Intuitive collisions 🌶️

Consider two particles (m_1,m_2) with velocities (v_1,v_2) before head-to-head collision. What will the situation be after collision, tell so without calculations, if:

1. $m_1 = m_2$ and $v_1 = v; v_2 = 0$
2. $m_1 = m_2$ and $v_1 = v; v_2 = -v$
3. $m_1 = 2m_2$ and $v_1 = v; v_2 = 0$
4. $2m_1 = m_2$ and $v_1 = v; v_2 = 0$
5. $m_1 = 2m_2$ and $v_1 = v; v_2 = -v$

Exercise 2.292: 🌶️

A particle of mass $3m$ and velocity $2v$ will collide with a particle of mass $2m$ and velocity $-3v$. The problem is 1-dimensional.

- The collision is elastic. Find the velocities of the masses after the collision.
- The collision is completely inelastic. Find the velocities of the masses after the collision.
- The restitution coefficient is: e=1/5. Find the velocities of the masses after the collision.

Exercise 2.293: 🌶️

A particle of mass $2m$ moves over the x-axis with velocity v. It will collide with a particle of mass m that moves over the y-axis also with velocity v. The collision is completely inelastic.

Find the velocity of the particles after the collision and calculate the loss of kinetic energy.

Exercise 2.294: 🌶️

A tennis ball is dropped from a height of 1m (with zero initial velocity) on the tennis court. The restitution coefficient is $\frac{1}{2}\sqrt{2}$. After how many bounces does the tennis ball no longer reach a height of 0.25m. Friction with the air can be ignored.

Exercise 2.295: 🌶️ 🌶️

In Hollywood films often one of the persons is shot. That person (whether dead, wounded or 'just fine' for the hero) is blown of its feet and may fly a meter or more backwards.

The shooter, however, does not fly or fall backwards.

1. Show that if the victim moves backwards significantly, then the shooter shoot do at least the same.
2. A bullet weighs several grams and may have a velocity of several hundred m/s. Estimate what the backward velocity of a victim is. For comparison: when we walk, our velocity is 1 to 2m/s. Conclusion?

Solution 2.296: Solution to Exercise 5

Given: the collision is completely inelastic. That means $e = 0$ or in words: after the collision the two particles stick together and move as one particle. Thus, we have only one unknown velocity after the collision.

The problem is 1-dimensional and we can solve it by requiring conservation of momentum:

$$\begin{split}

\text{before ;;;;;;} mv &+ M \cdot 0 = (m+M)U \text{ ;;;;;; after} \ \Rightarrow U &= \frac{m}{m+M} v

\end{split}$$

Solution 2.297: Solution to Exercise 7

- $3m$ has velocity $-2v$ and $2m$ has velocity $3v$
- Both particles have zero velocity.
- $3m$ has velocity $-2/5v$ and $2m$ has velocity $3/5v$.

Solution 2.298: Solution to Exercise 8

$$\vec{v}_{after} = \frac{2}{3}v\hat{x} + \frac{1}{3}v\hat{y} \tag{2.442}$$

$$\Delta E_{kin} = -\frac{2}{3}mv^2 \tag{2.443}$$

Solution 2.299: Solution to Exercise 9

After each bounce, the tennis ball reaches half of the height it had before the bounce. Thus after two bounces, the ball reaches 25cm and with the third bounce only 12.5cm.

Solution 2.300: Solution to Exercise 10

1. We can consider the shooting as a collision. Bullets don't bounce back, they penetrate a body. So the victim 'gains' maximum momentum if the bullet stays in the body. Then according to conservation of momentum, we have for this inelastic collision:

$$m_b v_b + M_v \cdot 0 = (m_b + M_v)U \tag{2.444}$$

Thus the velocity of the victim after being shot is:

$$U = \frac{m_b}{m_b + M_v} v_b \tag{2.445}$$

For the shooter a similar argument holds: before the shot, bullet & shooter have zero momentum. After firing, the bullet has velocity v_b. Thus conservation of momentum requires:

$$0 = m_b v_b + M_s U_s \tag{2.446}$$

and we find for the velocity of the shooter:

$$U_s = -\frac{m_b}{M_s} v_b \tag{2.447}$$

Conclusion: as the mass of the bullet is negligible compared to that of a person both shooter and victim have similar velocities. As their mass is comparable, it is clear: from a physics point of view, if the victim is blown backward, than also the shooter is.

1. From the above we get, using $m_b \approx 10 \cdot 10^{-3}$kg, $v_b \approx 500$m/s and $M_v \approx 70$kg:

$$U_v = \frac{m_b}{m_b + M_v} v_b \approx 7cm/s \tag{2.448}$$

That is much too little to 'knock' someone over. Hollywood is good at 'dramatic effects', not so good at physics.

2.7.6.4 Answers

2.8 Two Body Problem: Kepler revisited

Updated: 04 feb 2026 Newton must have realized that his analysis of the Kepler laws was not 100% correct. After all, the sun is not fixed in space and even though its mass is much larger than any of the planets revolving it, it will have to move under the influence of the gravitational force by the planets. Take for example, the sun and earth as our system. By the account of Newton's third law, the Earth also exerts a force on the Sun. Therefore, the Sun has to move as well; thus, we must revisit the Earth-Sun analysis and incorporate that the Sun isn't fixed in space.

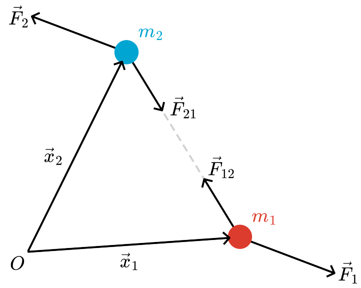


Figure 2.301: Two-particle system, with an action/reaction pair of forces.

The *two-body problem* is stated hereby as:

Particle m_1 feels an external force $\vec{F}_1$ and an interaction force from particle two, $\vec{F}_{21}$. Similarly for particle m_2: it feels an external force $\vec{F}_2$ and an interaction force from particle one, $\vec{F}_{12}$.

Consider the situation in the figure:

$$m_1\ddot{\vec{x}}_1 = \vec{F}_1 + \vec{F}_{21} \tag{2.449}$$

$$m_2\ddot{\vec{x}}_2 = \vec{F}_2 + \vec{F}_{12} \tag{2.450}$$

Add the two equations and use N3: $\vec{F}_{12} = -\vec{F}_{21}$:

$$m_1\ddot{\vec{x}}_1 + m_2\ddot{\vec{x}}_2 = \vec{F}_1 + \vec{F}_2 \Leftrightarrow \tag{2.451}$$

$$\dot{\vec{P}} = \vec{F}_1 + \vec{F}_2 \tag{2.452}$$

with $\vec{P} \equiv \vec{p}_1 + \vec{p}_2$. In words, it is as if a particle with (total) momentum $\vec{P}$ responds to the external forces but does not react to internal forces (the mutual interaction).

2.8.1 Center of Mass

It is now logical to assign the total mass $M = m_1 + m_2$ to this fictitious particle. It has momentum $\vec{p}_1 + \vec{p}_2$ which we can also couple to its mass M and assign a velocity $\vec{V}$ to it such that $\vec{P} = M\vec{V}$. Furthermore, if this fictitious mass has velocity $\vec{V}$, we can also assign a position to it. After all, $\vec{V} = \frac{d\vec{R}}{dt}$, which gives us the recipe for the position $\vec{R}$.

Its velocity $\vec{V}$ and position $\vec{R}$ then follow as:

$$\begin{aligned} \vec{V} &= \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} \\ &= \frac{m_1\frac{d\vec{x}_1}{dt} + m_2\frac{d\vec{x}_2}{dt}}{m_1 + m_2} \\ \Rightarrow \vec{R} &= \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2} + \vec{C} \end{aligned} \tag{2.453}$$

In the last equation, we have an integration constant in the form of a vector, $\vec{C}$. We are free to choose it as we want: its precise value does not affect the velocity $\vec{V}$ nor the momentum $\vec{P}$ of our fictitious particle.

It makes sense, to choose: $\vec{C} = 0$ and thus define as position of the particle:

$$\vec{R} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2} \tag{2.454}$$

Why?

We have a few arguments:

1. if the particles are actually each half of a real particle, we obviously require that $\vec{R}$ is the position of the real particle.
2. If the particles are separate by a small distance, we would like to have the fictitious particle somewhere in between the two. Moreover, if the two particles are identical, it makes sense to have the fictitious particle right in between them: the system is symmetric.

Where, in general is the position $\vec{R}$? That can be easily seen from the figure below.

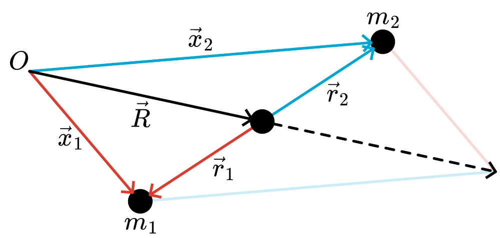


Figure 2.302: Center of Mass and relative coordinates.

We rewrite the definition of $\vec{R}$:

$$\vec{R} \equiv \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2} = \vec{x}_1 + \frac{m_2}{m_1 + m_2}(\vec{x}_2 - \vec{x}_1) \tag{2.455}$$

Thus, the last part of the above equation tells us: first go to m_1 and then, 'walk' a fraction $\frac{m_2}{m_1+m_2}$ of the line connecting m_1 and m_2. If you have done that, you are at position $\vec{R}$.
Note: if $m_1 = m_2$ this recipe indeed brings us right between the two particles.
Further note: the position of M is always on the line from m_1 to m_2. If m_1 is much larger than m_2, it will be located close to m_1 and vice versa.

We call this position the **center of mass**, or CM for short. Reason: if we look at the response of our two particle system to the forces, it is as if there is a particle M at position $\vec{R}$ that has all the momentum of the system.

It turns out to be convenient to define relative coordinates with respect to the center of mass position (see also the figure above):

$$\vec{r}_1 \equiv \vec{x}_1 - \vec{R} and \vec{r}_2 \equiv \vec{x}_2 - \vec{R} \tag{2.456}$$

Via the external forces, we can 'follow' the motion of the center of mass position, i.e. $\vec{R}$. From the CM as new origin, we can find the position of the two particles.

A helpful rule is found from:

$$\begin{gathered} m_1\vec{r}_1 + m_2\vec{r}_2 = \\ = m_1\left(\vec{x}_1 - \vec{R}\right) + m_2\left(\vec{x}_2 - \vec{R}\right) \\ = m_1\vec{x}_1 + m_2\vec{x}_2 - (m_1 + m_2)\vec{R} = 0 \end{gathered} \tag{2.457}$$

$$\Rightarrow m_1\vec{r}_1 + m_2\vec{r}_2 = 0 \tag{2.458}$$

This has an important consequence: if we know $\vec{r}_1$, we know $\vec{r}_2$, and vice versa. Note: the directions of $\vec{r}_1$ and $\vec{r}_2$ are always opposed and the center of mass $\vec{R}$ is located somewhere on the connecting line between m_1 and m_2.

Note 2: in the case of no external forces $\vec{F}_1 = \vec{F}_2 = 0$ and only internal forces $\vec{F}_{12} \neq 0$ the CM moves according to N1 with constant velocity $\left(\dot{\vec{P}} = 0\right)$.

2.8.2 Energy

In terms of relative coordinates, we can write the kinetic energy as a part associated with the CM and a part that describes the kinetic energy with respect to the CM, i.e., 'an internal kinetic energy.'

$$\begin{aligned} E_{kin} &\equiv \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 \\ &= \tfrac{1}{2}m_1\left(\dot{\vec{r}}_1 + \dot{\vec{R}}\right)^2 + \tfrac{1}{2}m_2\left(\dot{\vec{r}}_2 + \dot{\vec{R}}\right)^2 \\ &= \tfrac{1}{2}M\dot{\vec{R}}^2 + \tfrac{1}{2}m_1\dot{\vec{r}}_1^2 + \tfrac{1}{2}m_2\dot{\vec{r}}_2^2 \end{aligned} \tag{2.459}$$

For the potential energy, we may write:

$$V = \sum V_i + \frac{1}{2}\sum_{i\neq j}(V_{ij} + V_{ji}) \tag{2.460}$$

With V_i the potential related to the external force on particle i and V_{ij} the potential related to the mutual interaction force from particle i exerted on particle j (assuming that all forces are conservative).

2.8.3 Angular Momentum

The total angular momentum is, like the total momentum, defined as the sum of the angular momentum of the two particles:

$$\vec{L} = \vec{l}_1 + \vec{l}_2 = \vec{x}_1 \times \vec{p}_1 + \vec{x}_2 \times \vec{p}_2 \tag{2.461}$$

We can write this in the new coordinates:

$$\vec{L} = \vec{R} \times \vec{P} + \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = \vec{L}_{cm} + \vec{L}' \tag{2.462}$$

We find: that the total angular momentum can be seen as the contribution of the CM and the sum of the angular momentum of the individual particles as seen from the CM.

2.8.4 Reduced Mass

Suppose that there are no external forces. Then the equation of motion for both particles reads as:

$$\begin{aligned} m_1\ddot{\vec{x}}_1 &= \vec{F}_{12} \\ m_2\ddot{\vec{x}}_2 &= \vec{F}_{21} = -\vec{F}_{12} \end{aligned} \tag{2.463}$$

If we divide each equation by the corresponding mass and subtract one from the other we get:

$$\frac{d^2}{dt^2}(\vec{x}_1 - \vec{x}_2) = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}_{12} \tag{2.464}$$

Note that the interaction force $\vec{F}_{12}$ is a function of the relative position of the particles, i.e., $\vec{x}_1 - \vec{x}_2 = \vec{r}_1 - \vec{r}_2$.

Introduce $\vec{r}_{12} \equiv \vec{r}_1 - \vec{r}_2 = \vec{x}_1 - \vec{x}_2$, then we obtain:

$$\frac{d^2}{dt^2}\vec{r}_{12} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}_{12}(\vec{r}_{12}) \tag{2.465}$$

As a final step, we introduce the *reduced mass* μ:

$$\frac{1}{\mu} \equiv \frac{1}{m_1} + \frac{1}{m_2} \Leftrightarrow \mu = \frac{m_1 m_2}{m_1 + m_2} \tag{2.466}$$

And we can reduced the two-body problem to a single-body problem, by writing down the equation of motion for an imaginary particle with reduced mass.

$$\mu \frac{d^2\vec{r}_{12}}{dt^2} = \vec{F}_{12} \tag{2.467}$$

If $m_1 \gg m_2$ we have $\mu \to m_2$. This is not surprising: when m_1 is much larger than m_2, it will look like m_1 is not changing its velocity at all. Or seen from the CM: is appears to be not moving. In this case, we can ignore particle 1 and replace it by a force coming out of a fixed position.

2.8.4.1 Back to the Two-Body Problem

Once we solved the problem for the reduced mass, it is straightforward to go back to the two particles. It holds that:

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0 \tag{2.468}$$

$$\vec{r}_2 = -\frac{m_1}{m_2}\vec{r}_1 \qquad \vec{r}_2 = \vec{r}_1 - \vec{r}_{12} \tag{2.469}$$

$$\begin{array}{rcl} \vec{r}_1 & = & \frac{m_1}{m_1+m_2}\vec{r}_{12} \\ \vec{r}_2 & = & -\frac{m_1}{m_1+m_2}\vec{r}_{12} \end{array} \tag{2.470}$$

Thus, if we have solved the motion of the reduced particle, then we can quickly find the motion of the two individual particles (seen from the CM frame).

2.8.5 Kepler Revisited

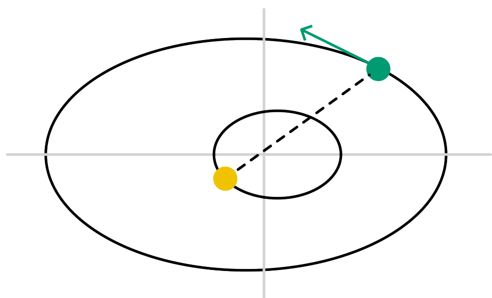


Figure 2.303: Kepler revisited.

Now that we have seen how to deal with the two-body problem, we can return to the motion of the Earth around the Sun. This is obviously not a two-body problem, but a many-body problem with many planets.

However, we can approximate it to a two-body problem: we ignore all other planets and leave only the Sun and Earth. Hence, there are no external forces. Consequently, the CM of the Earth-Sun system moves at a constant velocity. And we can take the CM as our origin.

We have to solve the reduced mass problem to find the motion of both the Earth and the Sun:

$$\mu \frac{d^2\vec{r}_{12}}{dt^2} = -\frac{Gm_e m_s}{r_{12}^2}\hat{r}_{12} \tag{2.471}$$

Note: this equation is almost identical to the original Kepler problem. All that happened is that m_e on the left hand side got replaced by μ.

Everything else remains the same: the force is still central and conservative, etc.

2.8.5.1 Where is the CM located?

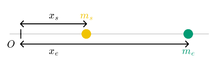


Figure 2.304: Position of CM in the sun-earth system.

We can easily find the center of mass of the Earth-Sun system. Chose the origin on the line through the Sun and the Earth (see fig.)

$$R = \frac{m_s x_s + m_e x_e}{m_s + m_e} = x_s + \frac{m_e}{m_s + m_e}(x_e - x_s) \approx x_s + 450km \tag{2.472}$$

In other words: the Sun and Earth rotate in an ellipsoidal trajectory around the center of mass that is 450 km out of the center of the Sun. Compare that to the radius of the Sun itself: $R_s = 7 \cdot 10^5$ km. No wonder Kepler didn't notice. The common CM and rotation point is called Barycenter in astronomy.

2.8.5.2 Exoplanets

However, in modern times, this slight motion of stars is a way of trying to find orbiting planets around distant stars. Due to this small ellipsoidal trajectory, sometimes a star moves away from us, and sometimes it comes towards us. This moving away and towards us changes the apparent color of the light emission of molecules or atoms by the Doppler effect. This is a periodic motion, which lasts a 'year' of that solar system. Astronomers started looking out for periodic changes in the apparent color of the light of stars. One can also look for periodic changes in the brightness of a star (which is much, much harder than looking at spectral shifts of the light). If a planet is directly between the star and us, the intensity of the starlight decreases a bit. And they found one, and another one, and more and hundreds... Currently, more than 5,000 exoplanets have been found.

Figure 2.305: Changing color of star light due to a period motion induced by a planet orbiting the star (animation from NASA).

Figure 2.306: Changing intensity of star light due to a period passage of a planet orbiting the star (animation from NASA).

2.8.6 Many-Body System

We have seen that we could reduce the two-body problem of sun-earth to a single body question via the concept of reduced mass. But that this strategy does not work for 3, 4, 5, ... bodies.

2.8.6.1 Linear Momentum

We can, however, find some basic features of N-body problems. In the figure, a collection of N interacting particles is drawn.

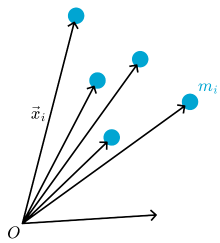


Figure 2.307: Many particle system.

Each particle has mass m_i and is at position $x_i(t)$. For each particle, we can set up N2:

$$m_i \ddot{\vec{x}}_i = \vec{F}_{i,ext} + \sum_{i \neq j} \vec{F}_{ji}. \tag{2.473}$$

Summing over all particles and using that all mutual interaction forces form “action = - reaction pairs”, we get:

$$\sum_i m_i \ddot{\vec{x}}_i = \sum_i \vec{F}_{i,ext} \Leftrightarrow \sum_i \dot{\vec{p}}_i = \sum_i \vec{F}_{i,ext} \tag{2.474}$$

The second part can be written as:

$$\frac{d\vec{P}}{dt} = \sum_i \vec{F}_{i,ext} with \vec{P} \equiv \sum_i \vec{p}_i \tag{2.475}$$

In other words: the total momentum changes due to external forces. If there are no external forces, then the total momentum is conserved. This happens quite a lot actually, if you consider e.g. collisions or scattering.

2.8.6.2 Center of Mass

Analogous to the two-particle case, we see from the total momentum that we can pretend that there is a particle of total mass $M = \sum_i m_i$ that has momentum $\vec{P}$, i.e., it moves at velocity $\vec{V} \equiv \frac{\vec{P}}{M}$ and is located at position:

$$\vec{V} = \frac{d\vec{R}}{dt} = \frac{\sum m_i \frac{d\vec{x}_i}{dt}}{\sum m_i} \Rightarrow \vec{R} = \frac{\sum m_i \vec{x}_i}{\sum m_i} \tag{2.476}$$

Continuing with the analogy, we define relative coordinates:

$$\vec{r}_i \equiv \vec{x}_i - \vec{R} \tag{2.477}$$

and have a similar rule constraining the relative positions:

$$\sum m_i \vec{r}_i = 0 \tag{2.478}$$

2.8.6.3 Energy

In terms of relative coordinates, we can write the kinetic energy as a part associated with the center of mass and a part that describes the kinetic energy with respect to the center of mass, i.e., ‘an internal kinetic energy’.

$$\begin{aligned} E_{kin} &\equiv \sum \frac{1}{2} m_i v_i^2 \\ &= \frac{1}{2} M \dot{\vec{R}}^2 + \sum \frac{1}{2} m_i \dot{\vec{r}}_i^2 \\ &= E_{kin,cm} + E'_{kin} \end{aligned} \tag{2.479}$$

For the potential energy, we may write:

$$V = \sum V_i + \frac{1}{2} \sum_{i \neq j} \left(V_{ij} + V_{ji}\right) \tag{2.480}$$

with V_i the potential related to the external force on particle i and V_{ij} the potential related to the mutual interaction force from particle i exerted on particle j (assuming that all forces are conservative).

2.8.6.4 Angular Momentum

The total angular momentum is, like the total momentum, defined as the sum of the angular momentum of all particles:

$$\vec{L} = \sum \vec{l}_i = \sum \vec{x}_i \times \vec{p}_i \tag{2.481}$$

We can write this in the new coordinates:

$$\vec{L} = \vec{R} \times \vec{P} + \sum \vec{r}_i \times \vec{p}_i = \vec{L}_{cm} + \vec{L}' \tag{2.482}$$

Again, we find that the total angular momentum can be seen as the contribution of the center of mass and the sum of the angular momentum of all individual particles as seen from the center of mass.

The N-body problem is, of course, even more complex than the three-body problem. If we can solve it, it will be under very specific conditions only. However, a numerical approach can be done with great success. Moreover, current computers are so powerful that the system can contain hundred, thousands of particles up to billions depending on the type or particle-particle interaction.

All kind of computational techniques have been developed and various averaging techniques are employed in statistical techniques are introduced from the start. the reason is often, that a particular ‘realisation’ of all positions and velocities of all particles is needed nor sought for. A system is at its macro level described by averaged properties, the exact location of the individual atoms is not needed. You will find applications in cosmology all the way to molecular dynamics, trying to simulate the behavior of proteins or pharmaceuticals.

2.8.7 Three body Problem

Now that we have reduced a two-particle system to a single particle problem, the question arises: can we repeat this 'trick' and turn a three-body problem in a two body problem, that in its turn can be reduced to a single particle problem?

The answer is: no. There is no general strategy to reduce a three body problem two a two body-one.

The three body problem is an old one. Already Newton himself worked on it. Its importance stems e.g. from navigation on sea. It would be of great help if the position of the moon could be predicted in advance with great accuracy. Then sailors in the 17^{th}, 18^{th} and 19^{th} could have found much better their position at full sea. But no one succeeded in providing a closed solution in basic functions.

The king of Sweden, Oscar II, announced, as celebration of his 60^{th} birthday, a competition with the price awarded to the one that came up with a general solution. But it took a different course. The price went to the French mathematician and physicist Henri Poincaré.

Figure 2.308: Click here for the Wikipedia page of Poincaré.

He showed that it was impossible to find such a solution as he reached the conclusion that the three body problem showed chaotic features. It led Poincaré to develop a whole new field: dynamic systems and what we call now *deterministic chaos.*
The work of Poincaré was the trigger of yet another 'revolution' in our understanding of the universe.

It doesn't mean that there are no known solutions of specific cases of the three body problem. On the contrary, in the animation below 20 solutions are given. Notice that they all have a high degree of symmetry.

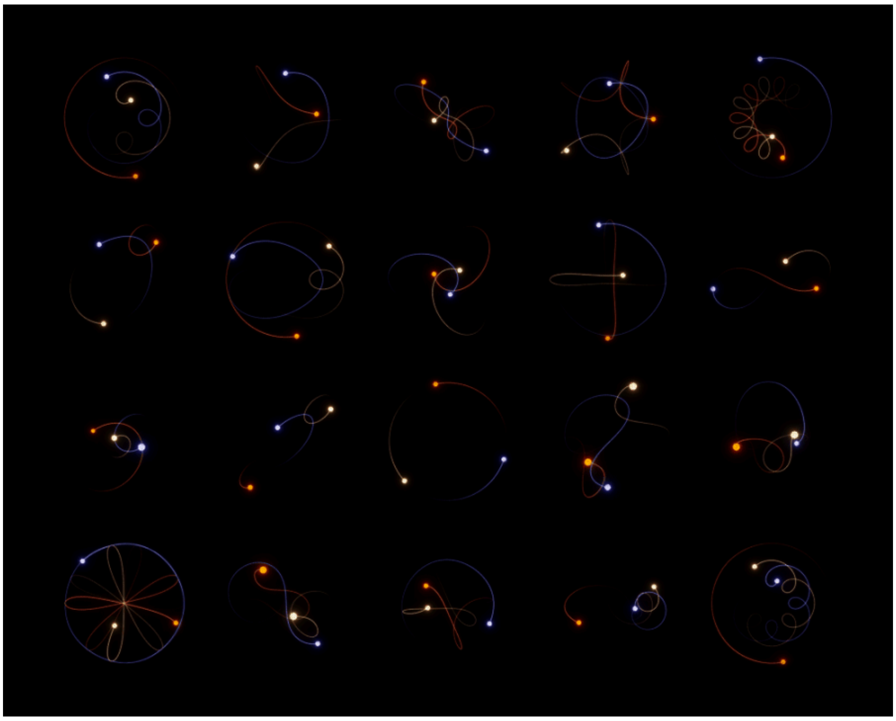


Figure 2.309: Click here to see some exact solutions of the three body problem (By Perosello - Uploaded by Author, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=133294338).

2.8.7.1 Alpha Centauri A, Alpha Centauri B and Alpha Centauri C

The three body problem can also be studied by numerical means. As the equations of motion are easily set up and put into a computer code, this allows us to investigate for instance the three stars of the Alpha Centauri system: Alpha Centauri A, Alpha Centauri B and Alpha Centauri C. This system is a little over 4 million light years away from us: these stars are our closest (star) neighbors. Although they form a three body system, it is stable due to the much smaller mass op Alpha Centauri C compared to the other two. Alpha Centauri A and Alpha Centauri B are of similar mass, that is 1.1 and 0.9 the mass of our sun, respectively. Alpha Centauri C, on the other hand has a mass of only 0.12 of that of the sun.

Gaurav Deshmukh has written a nice python-based web-page on this system. Below we show some examples of the simulations, that you can do yourself with the code given by Deshmukh.

First, we ignore Alpha Centauri C and used that A and B have about the same mass. The two stars start rotating around each other in ellipsoidal orbits, as we already know from the two body problem.

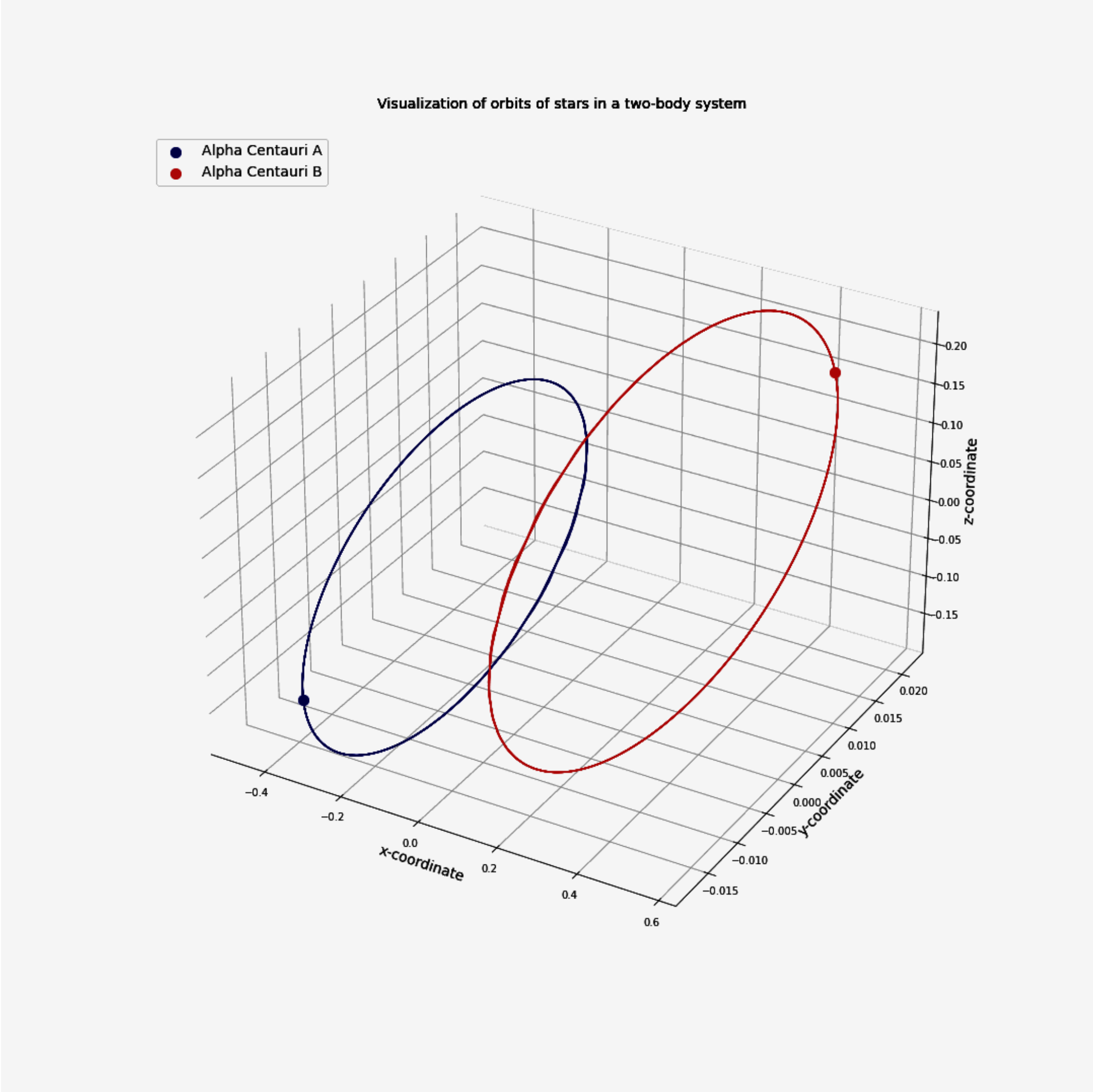


Figure 2.310: Alpha Centauri A and B circling each other.

Then, we add third small one object (not Centauri C, but one with a much smaller mass): m_A = 1.1, m_B = 0.907 (both actual relative masses), m_C = 0.001.

m_C tries to orbit its closest star, but at some point comes under the influence of the second star and gets ‘tossed around’.

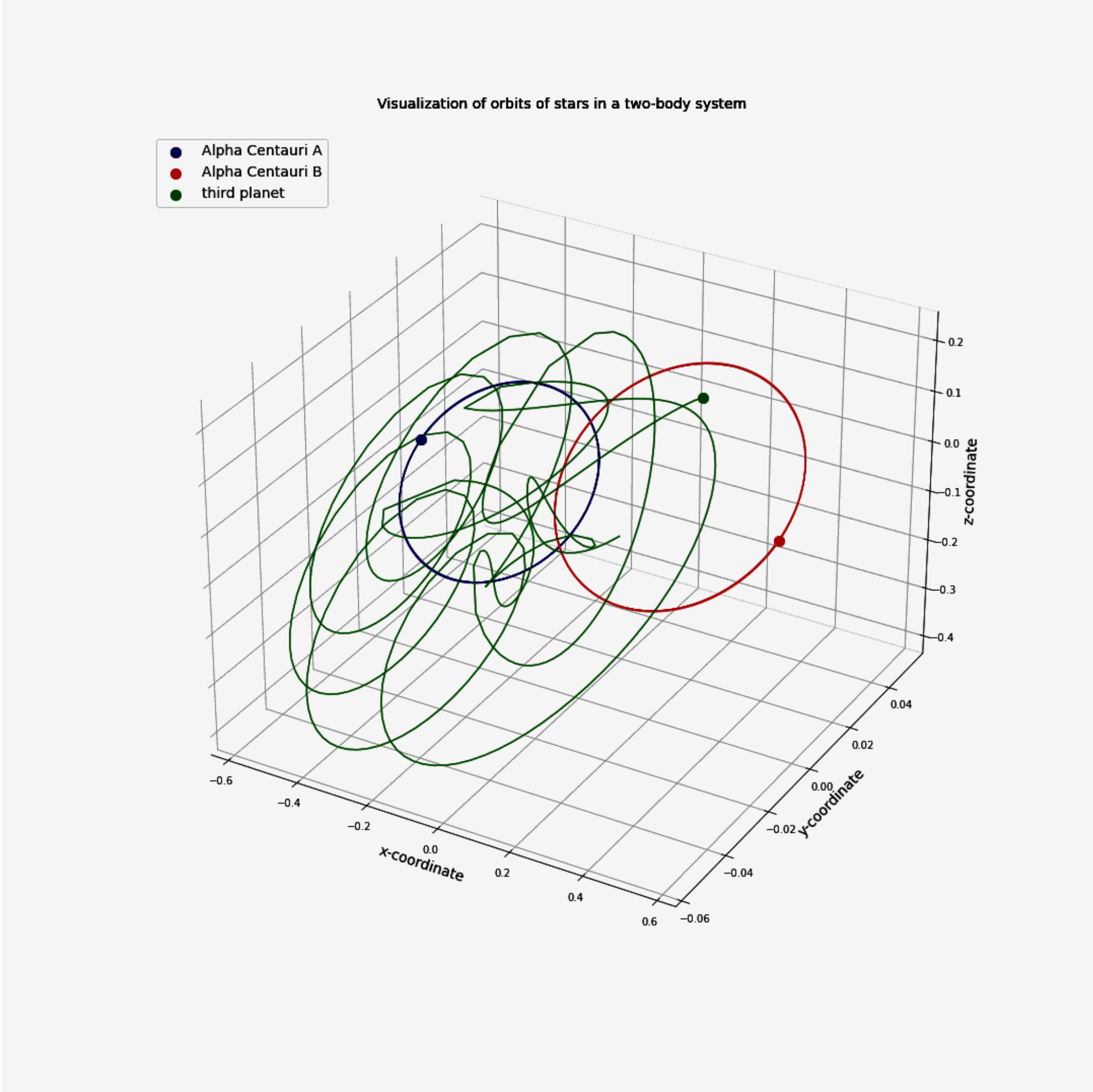


Figure 2.311: Alpha Centauri A and B circling each other with a third object.

If we let the simulations run for a much longer time, we see that at some point the conditions for our small star are such that it is ‘shot’ into space and disappears for ever.

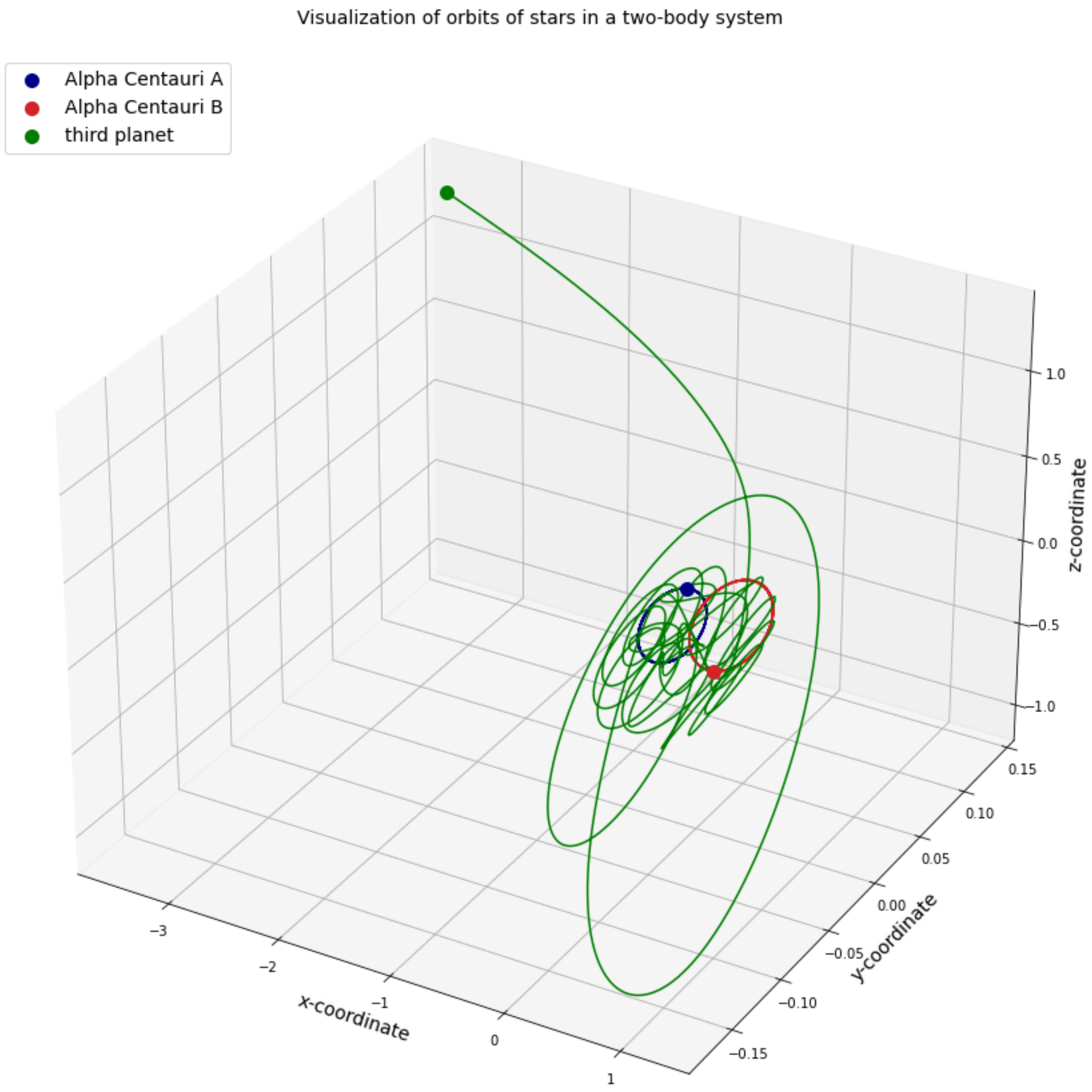


Figure 2.312: Alpha Centauri A and B circling each other with a third object. The third 'planet' is finally escaping into space.

Note: this is a chaotic system and computations need great care.

Figure 2.313: A stable solution of the three body problem, but slightly change one of the parameters and the solution is not stable anymore!

Three body problem

NetFlix has a great tv series called 3 Body Problem

2.8.8 Examples, exercises & solutions

Updated: 04 feb 2026

2.8.8.1 Worked examples

2.8.8.1.1 Title of example

Interpret the problem

HIER DE INTERPRETATIE

Develop the solution

HIER DE DEVELOPMENT

Evaluate the problem

HIER DE EVALUATE

Assess the problem

HIER DE ASSESS

Exercise 2.314: 🌶️

In the table below, the mass and distance from the sun of the planets in our solar system are given (in terms of the earth mass and distance from the earth to the sun). Compute for each planet-sun pair the distance from the center of mass to the center of the sun. Given: distance CM to center of sun for the earth-sun system is 450km.

planet	relative mass	relative distance to the sun
Mercurius	0.06	0.39
Venus	0.82	0.72
Earth	1.00	1.00
Mars	0.11	1.52
Jupiter	317.8	5.20
Saturnus	095.2	9.54
Uranus	14.6	19.22
Neptunus	17.2	30.06

Exercise 2.315: 🌶️

Two particles $m_1 = m$ and $m_2 = 2m$ are traveling both along the x-axis. At $t = 0$ the particles have both velocity $v_0 > 0$. Their positions at $t = 0$ are $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$ with $x_{10} < x_{20}$. They repel each other with a force $F_r = \frac{k}{(x_2 - x_1)^2}$. Moreover, a constant external force F_e is acting on them. The problem is 1-dimensional.

- Find the velocity of the center of mass for $t > 0$
- Find the position of the center of mass for $t > 0$.

Exercise 2.316: 🌶️

Two particles $m_1 = 3\mathrm{kg}$ and $m_2 = 2\mathrm{kg}$ are connected via a massless rod of length $L = 50\mathrm{cm}$.

- Find the position of the center of mass of the system, measured from m_1
- Calculate the reduced mass of the two-particle system.

Exercise 2.317: Bumper car collision 🌶️

Two bumper cars are approaching each other in a straight line. The two cars will collide head-on. The mass of car 1 (including the driver) is 200kg, that of car 2 300kg. Car 1 has a velocity of $8\mathrm{m/s}$; $car 2 of -4$ \mathrm{m/s}$.

- What is the velocity of the center of mass of the system?
- What is the reduced mass of the system?
- Transform the velocities of both carts to the center-of-mass frame.

Exercise 2.318: 🌶️

Two carts on a frictionless track move toward each other:

Cart 1: mass $m_1 = 2\mathrm{kg}$, velocity $v_1 = 4\mathrm{m/s}$
Cart 2: mass $m_2 = 3\mathrm{kg}$, velocity $v_2 = -2\mathrm{m/s}$

- What is the total kinetic energy in the lab frame?
- What is the velocity of the center of mass?
- What is the total kinetic energy in the center-of-mass frame?
- Verify that the CM frame kinetic energy equals the kinetic energy due to relative motion using the reduced mass.

Solution 2.319: Solution to Exercise 1

planet	relative mass	relative distance to the sun	distance CM to center of sun (km)
Mercurius	0.06	0.39	10
Venus	0.82	0.72	265
Earth	1.00	1.00	450
Mars	0.11	1.52	75
Jupiter	317.8	5.20	$743 \cdot 10^3$
Saturnus	095.2	9.54	$409 \cdot 10^3$
Uranus	14.6	19.22	$126 \cdot 10^3$
Neptunus	17.2	30.06	$234 \cdot 10^3$

Solution 2.320: Solution to Exercise 2

We set up the equation of motion for the particles:

$$\begin{aligned} m_1 &: m_1\dot{v}_1 = F_e - F_r \\ m_1 &: m_2\dot{v}_2 = F_e + F_r \end{aligned} \tag{2.483}$$

Add these two equations:

$$M\dot{V} = m_1\dot{v}_1 + m_2\dot{v}_2 = 2F_e \to \dot{V} = \frac{2F_e}{m_1 + m_2} = \frac{2F_e}{3m} \tag{2.484}$$

As expected, we see that the repelling mutual force has no effect on the center of mass. We can solve this equation, using the initial condition the $MV(0) = m_1 v_1(0) + m_2 v_2(0) \to V(0) = \frac{mv_0 + 2mv_0}{m+2m} = v_0$

$$V(t) = \frac{2F_e}{3m}t + C_1 = \frac{2F_e}{3m}t + v_0 \tag{2.485}$$

As the next step we calculate $R(t)$:

$$\dot{R} \equiv V = v_0 + \frac{2F_e}{3m}t \to R(t) = v_0 t + \frac{F_e}{3m}t^2 + C_2 \tag{2.486}$$

The initial condition is: $R(0) = \frac{m_1 x_1(0) + m_2 x_2(0)}{m_1 + m_2} = \frac{1}{3}x_{10} + \frac{2}{3}x_{20}$.

This gives

$$R(t) = \frac{1}{3}x_{10} + \frac{2}{3}x_{20} + v_0 t + \frac{F_e}{3m}t^2 \tag{2.487}$$

Solution 2.321: Solution to Exercise 3

The center of mass of two point masses is on the line connecting m_1 and m_2. We denote this line as the x-axis, with m_1 as the origin.

- The center of mass is than given by (with m_1 = 3kg, m_2 = 2kg, x_1=0 and $x_2 = x_1 + L$ = 0.5m):

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = 0.2m \tag{2.488}$$

- The reduced mass is given by:

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} = \frac{6}{5}kg \tag{2.489}$$

Solution 2.322: Solution to Exercise 4

This is a 1-dimensional problem.

- The velocity of the center of mass is:

$$V_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{4}{5} m/s \tag{2.490}$$

- The reduced mass is given by:

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} = 120kg \tag{2.491}$$

- In the CM frame the velocities of the cars are:

$$\begin{aligned} {v_1}' &= v_1 - V_{cm} = 7.2m/s \\ {v_2}' &= v_2 - V_{cm} = -4.8m/s \end{aligned} \tag{2.492}$$

Solution 2.323: Solution to Exercise 5

Cart 1: mass m_1 = 2kg, velocity v_1 = 4m/s
Cart 2: mass m_2 = 3kg, velocity v_2 = −2m/s

- The total kinetic energy in the lab frame is

$$E_{kin} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = 22J \tag{2.493}$$

- The velocity of the center of mass is

$$V_{cm} \equiv \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = 0.4m/s \tag{2.494}$$

- The total kinetic energy in the center-of-mass frame is

$$E_{kin,CM} = \frac{1}{2}m_1 {v_1}'^2 + \frac{1}{2}m_2 {v_2}'^2 \tag{2.495}$$

with

$$\begin{aligned} {v_1}' &= v_1 - V_{cm} = 3.6m/s \\ {v_2}' &= v_2 - V_{cm} = -2.4m/s \end{aligned} \tag{2.496}$$

Thus

$$E_{kin,CM} = 21.6J \tag{2.497}$$

- The reduced mass is

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} = 1.2kg \tag{2.498}$$

The relative velocity is

$$v_{rel} \equiv v_1 - v_2 = 6m/s \tag{2.499}$$

The kinetic energy associated with the motion of the reduced mass (i.e. the kinetic energy in the CM frame) is:

$$E_{kin,rel} \equiv \frac{1}{2}\mu v_{rel}^2 = 21.6J \tag{2.500}$$

as we expected.

Exercise 2.324: 🌶️

Thee masses are forming an equilateral triangle with sides of 2m. Mass 1 (10kg) is positioned at $(x_1, y_1) = (-1m, 0))$. Mass 2 (6kg) is at $(x_2, y_2) = (1m, 0)$, while mass 3 (2kg) is at $(x_3, y_3) = \left(0, \sqrt{3}\right)$.

- Calculate the position of the center of mass.

Exercise 2.325: 🌶️ 🌶️

Four particles are moving over the line $y = y_0$. The particles have mass $m_1 = 4m, m_2 = 3m, m_3 = 2m, m_4 = m$ and velocity $v_1 = v, v_2 = 2v, v_3 = 3v, v_4 = 4v$. These velocities are constant and parallel to the x-axis. At $t = 0$ all particles are at location $(x, y) = (0, y_0)$.

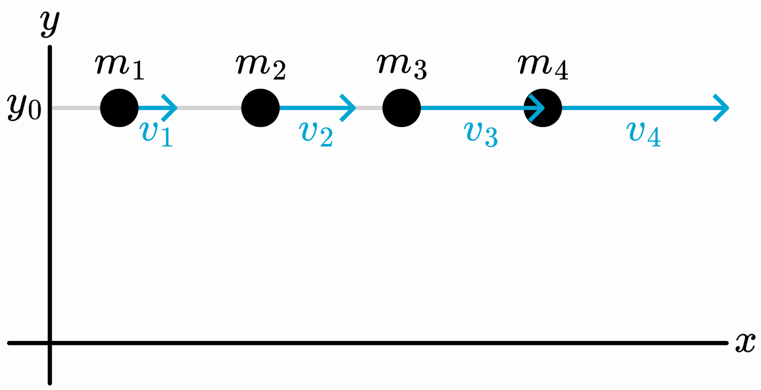


Figure 2.326: Four particles moving on a line.

- Calculate the velocity of the center of mass.
- Calculate the position of the center of mass as a function of time.
- Calculate the total angular momentum.
- Calculate the angular momentum associated with the center of mass and show that in this case this is equal to the total angular momentum.

Exercise 2.327: 🌶️ 🌶️

Eight point particles (each mass m) are attached to a massless wheel of radius R.

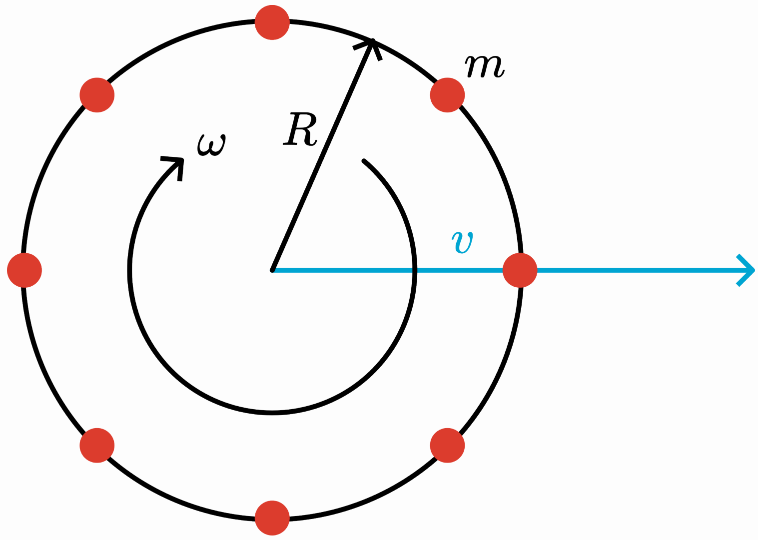


Figure 2.328: Eight particles on a wheel.

The wheel is moving with a velocity V while it is rotating at the same time with angular velocity ω.

Calculate the total kinetic energy of this system. Hint: use the CM frame and connect that to the lab frame.

Exercise 2.329: 🌶️ 🌶️ 🌶️

A container of volume V_c and mass M_c contains Nitrogen gas. The number of molecules, N, is on the order of 10^{23}. The container is dropped from a height H. Gravity is acting on the molecules. Friction on the container is ignored.

Show that the container falls with the acceleration of gravity g.

Exercise 2.330: 🌶️ 🌶️ 🌶️

We consider a 2-dimensional problem: $N = 30$ particles move in the xy-plane. Each particle has a fixed velocity $\left(v_x^i, v_y^i\right)$ with $i = 1..N$. The particle velocities have a magnitude ranging from 1 to 5 (m/s) randomly chosen for each particle. The direction of each velocity vector is also randomly chosen from 0 to 2π. The particles move inside a box with sides L=50m. Particles do not collide with each other, but they do collide with the walls of the container. The result of a collision is that the particle motion gets reflected.

- Write a python program that generates N particles starting all at $(x, y) = (0, 0)$.
- Compute the position of all particles after 1 second and compute the velocity and position of the center of mass.
- Write a loop that updates the particle velocities after a time step dt and recompute the velocity and position of the center of mass.
- Run the loop M times and plot the position of the center of mass in the xy-plane as a function of time.
- What happens if you change the number of particles from 30 to 3 or to 300?

Solution 2.331: Solution to Exercise 6

The position of the center of mass is

$$\vec{R} \equiv \frac{\sum_i m_i \vec{x}_i}{\sum_i m_i} = \frac{(m_1 x_1)\hat{x} + (m_2 x_2)\hat{x} + (m_3 y_3)\hat{y}}{m_1 + m_2 + m_3} = -\frac{2}{9}[m]\hat{x} + \frac{1}{9}[m]\hat{y} \quad (2.501)$$

where $[m]$ indicates that the unit is meters.

Note: $\hat{x}$ and $\hat{y}$ do not carry units!

2.8.8.1.2 Exercises

2.8.8.1.3 Answers

2.8.8.1.4 Exercises

2.8.8.1.5 Answers

Solution 2.332: Solution to Exercise 7

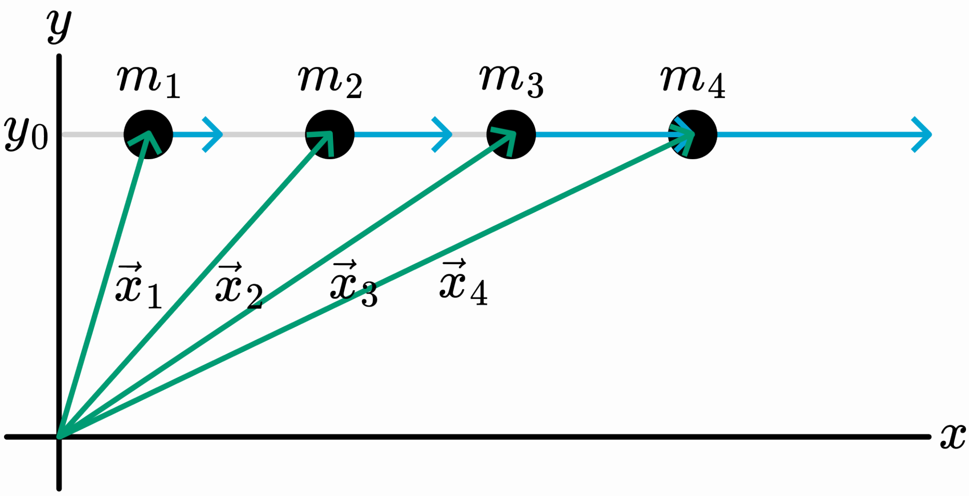


Figure 2.333:

2.8.8.2 width: 350px align: center

Four particles moving on a line.

- Velocity of the center of mass:

$$\vec{V} = \frac{\sum_i m_i \vec{v}_i}{\sum_i m_i} \tag{2.502}$$

Since the velocities are all parallel to the x-axis, we can drop the vector notation. Substituting the data for mass and velocity, gives:

$$V_x = \frac{4mv + 6mv + 6mv + 4mv}{4m + 3m + 2m + m} = 2v \tag{2.503}$$

- Position of the center of mass:

$$\vec{V} = \frac{d\vec{R}}{dt} \rightarrow \vec{R}(t) = 2vt\hat{x} + \vec{c} \tag{2.504}$$

At $t = 0$ all particles at location $(0, y_0)$. Thus, we find

$$\vec{R}(t) = 2vt\hat{x} + y_0\hat{y} \tag{2.505}$$

- Total angular momentum:

$$\begin{aligned} \overrightarrow{L_{tot}} &= \sum_i \vec{l}_i \\ &= y_0 \cdot 4mv\hat{z} + y_0 \cdot 3m \cdot 2v\hat{z} + y_0 \cdot 2m \cdot 3v\hat{z} + y_0 \cdot m \cdot 4v\hat{z} \\ &= 20mvy_0\hat{z} \end{aligned} \tag{2.506}$$

- Angular momentum associated with the center of mass:

$$\vec{L} = \vec{R} \times M\vec{V} = y_0 10m \cdot 2v\hat{z} = 20mvy_0\hat{z} \tag{2.507}$$

which is indeed the same as the total angular momentum. This is in this case to be expected as the angular momentum seen from the CM frame is $\vec{L}' = 0$ as in the CM frame the position vector and momentum vector are parallel for all four particles.

Solution 2.334: Solution to Exercise 8

We split the kinetic energy in the kinetic energy associated with the center of mass and the kinetic energy as seen from the CM frame:

$$E_{kin} = \frac{1}{2}MV^2 + E'_{kin} \tag{2.508}$$

Due to symmetry, the center of mass velocity is V.

In the CM frame, all particles rotate with ω and thus have a velocity of magnitude $v' = \omega R$. As all particles have the same mass, we have $M = 8m$. The kinetic energy is:

$$E_{kin} = \frac{1}{2}8V^2 + 8 \cdot \frac{1}{2}m\omega^2 R^2 = 4mV^2 + 4mR^2\omega^2 \tag{2.509}$$

Solution 2.335: Solution to Exercise 9

All nitrogen molecules feel gravity and have interaction with each other and with the wall of the container. If we write down the equation of motion for all molecules (labelled i) and the container we get:

$$\begin{aligned} M_c\ddot{\vec{x}}_c &= M_c\vec{g} + \sum_i \vec{F}_{molecule\ i\ on\ vessel\ wall} \\ m_i\ddot{\vec{x}}_i &= -m_i\vec{g} + \vec{F}_{vessel\ wall\ on\ molecule\ i} + \sum_{j\neq i}\vec{F}_{ji} \end{aligned} \tag{2.510}$$

with $\vec{F}_{molecule\ i\ on\ vessel\ wall}$ the force of molecule i on the vessel wall and $\vec{F}_{ji}$ the force from molecule j on molecule i. All these forces are internal forces and when summing over all particles (including the vessel) will cancel each other as they all obey N3.

Thus is we add the equations, we find:

$$\frac{d}{dt}\left(M_c\dot{\vec{x}}_c + \sum_i m_i\dot{\vec{x}}_i\right) = \left(M_c + \sum m_i\right)\vec{g} \tag{2.511}$$

On the left side, we recognize the total momentum which we can write in terms of the center of mass: $M_c\dot{\vec{x}}_c + \sum_i m_i\dot{\vec{x}}_i = M\vec{V}$.

And on the right hand side we see the total mass $M = M_c + \sum m_i$.

Thus, we conclude:

$$M\dot{\vec{V}} = M\vec{g} \to \dot{V} = -g \tag{2.512}$$

The entire container drops with acceleration $-g$.

Solution 2.336: Solution to Exercise 10

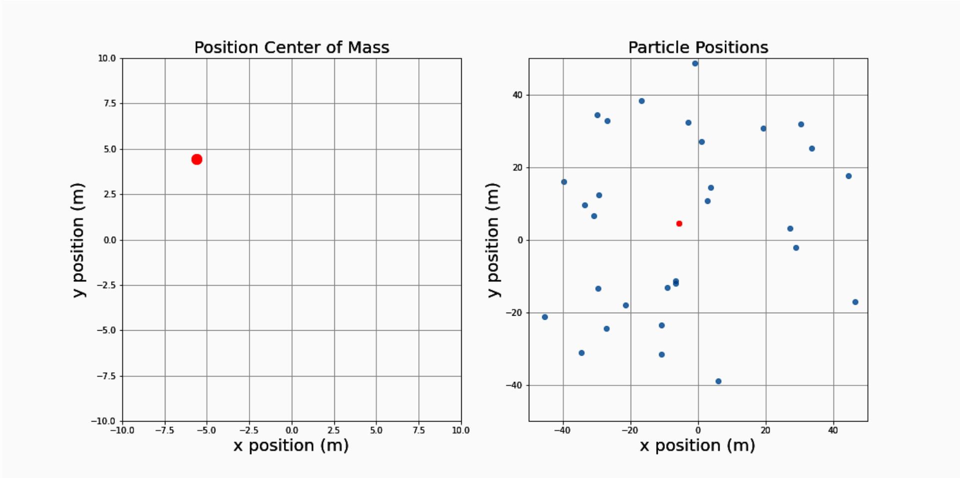


Figure 2.337:

2.8.8.3 :label: fig:Dustparticles_animation.gif width: 350px align: center

30 particles: left motion of the center of mass, right motion of all particles.

3. Special Relativity

Updated: 04 feb 2026 Instead of including rigid bodies, we opted for Special Relativity as part of the course. In our opinion, it is very stimulating for first-year students to dive into the fascinating world of Einstein and his Theory of Relativity. Moreover, Special Relativity is mathematically “light,” but it has strange, counterintuitive consequences. Through these, we can show that it is truly beneficial to work with a solid framework and to follow a rigorous approach when dealing with questions and problems. It helps students leave the high school approach—which is understandably a bit sloppy—behind and enter the world of physics.

We discuss the consequences of the fact that the speed of light in vacuum is the same for all inertial observers. We restrict ourselves to kinematics and collisions; therefore, four-forces and four-acceleration are not covered in this course. Length contraction, time dilation, and velocity transformation, however, are included. Moreover, we introduce the students to four-vectors that will help in understanding the new physics. We apply these concepts to collisions at high speeds and high energies. It goes without saying that we also do not treat General Relativity, as that requires a mastery of advanced mathematics.

It is always a joy to see when students are first confronted with Special Relativity: both confused and a bit “but this cannot be,” yet at the same time excited about this new world. The lecture breaks are then filled with students approaching us, sometimes confused, struggling to understand and appreciate the “weird” consequences, but always eager to learn. We wish for all physics students to become engaged and introduced to this new world as soon as possible.

3.1 Cracks in Classical Mechanics

Updated: 04 feb 2026 As the years progressed, Classical Mechanics developed further and further. So, in the first half of the nineteenths century it felt like classical mechanics was an all encompassing theory and that physics would become a discipline of working out problems based on a well-established, complete theory. But that wasn't going to be the case at all. Around 1850-1860 several cracks in the theory started to become visible. And they were fundamental!

3.1.1 Rutherford & the atom

3.1.1.1 Atomic theory

The idea that matter is made of atoms is old. However, the notion of atoms as we have now is relatively young.

In the ancient Greek world, it was as early as the fifth century B.C. that Leucippus and later one of his pupils Democritus proposed that the world (matter), is made up of tiny, indivisible particles. These particles were called atoms, derived from the Greek word 'atomos', which means uncuttable. These particles would float in a vacuum, that was called *void* by Democritus. We currently have a view that is remarkably close, but at the same time quite different from these first ideas.

In ancient India even earlier (records go back to the eighth' century B.C.) philosophers like Uddalaka Aruni talk about matter being made up of tiny particles. They did not use terms like atoms, but instead referred to the 'building blocks' of matter as 'kana' which means particles. In the Islamic world, atomic theories were developed in e.g. the Asharite school by Al Ghazali (1058-1111). In his thinking, atoms are the only material things that live forever. Everything else, any event or interaction is due to God's intervention.

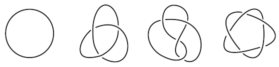
Although these early thoughts point at atoms as the underlying elements of matter and as such resemble our current understanding of matter, they also differ quite a bit. The early ideas are based on philosophy and the notion that matter is either a continuum that can always be cut in smaller parts that still maintain all characteristics or that at some point a further splitting in smaller pieces is no longer possible with at least losing some of the characteristics.

In more recent history, the notion of atoms as elementary building blocks is guided by experiments. The English physicist and chemist John Dalton (1766-1844) did ground breaking work. He noticed that water, when decomposed, always resulted in the same elements: hydrogen and oxygen. Moreover, the relative weights of these two was always the same. Furthermore, he came to the conclusion that there is a unique atom for each element. More chemists noted that many substances were made of the elements in very specific ratios. In our modern view we would say: water is formed in a 1 to 2 ratio of oxygen and hydrogen, never in 1 to 2.1 or any other non-integer number.

Although the evidence from chemistry was clear, the notion of atoms as the building blocks remained controversial. The laws of definite proportion (e.g. water is decomposed in a fixed ratio in hydrogen and oxygen) were generally accepted, but the hypothesis that everything was made of atoms was not. As a consequence, the work of Ludwig Boltzmann (1844-1906) on statistical thermodynamics that is entirely based on an atomic (or molecular) view was not accepted during Boltzmann's life.

In the second half of the nineteenth century William Thomson (1824-1907)—later Lord Kelvin—proposed the so-called vortex theory of the atom. Based on the discoveries by chemists of only a few different atoms that made up the rest of matter, Thomson proposed that atoms are stable vortices, not in an ordinary fluid like water, but in the omni-present luminiferous aether (ether).

Stable vortices have the shape of rings with no beginning or end. In air they are easily made and made visible with smoke and are indeed surprisingly stable. According to the vortex theory, atoms are vortices in aether. The simplest one is a single ring, which was hydrogen. More complicated forms, called knots represented the other elements.


(a)

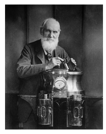
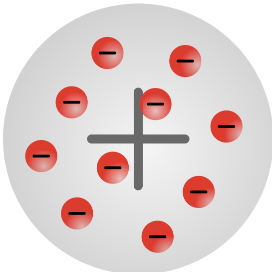

(b)

Figure 162: Lord Kelvin working on the vortex theory of the atom

At the end of the nineteenth century, in 1897, Joseph John Thomson discovered the electron. It allowed him to further refine the scientific model of the atom and ended the vortex theory. In Thomson's view, an atom has internal structure: the electrons are moving in it. As electrons have a negative charge and atoms are neutral, there must be a balancing positive charge in an atom as well. Thomson had no idea what that would be. He figured that the positive charge was everywhere in the atom (that he thought of as being a sphere), with the electrons moving inside that sphere as tiny particles. From this picture, the Thomson model got its name: *the plum pudding model*, although it is a bit misleading as the idea was that the positively charged sphere was more a liquid in which the electrons 'float' than a solid.


(a)

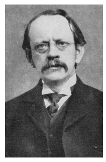

(b)

Figure 163: Thomson refined the idea of the atomic model: electrons moving within the atom, though the atom is still neutral in charge.

The model did not hold very long as we will see in the next paragraph. Nevertheless, it marks the start of physicist becoming really interested in an atom theory.

3.1.1.2 Rutherford's scattering experiment

The plum pudding model was abandoned in 1911. That year Ernest Rutherford (1871-1937), a former student of Joseph Thomson, performed a ground-breaking experiment. Rutherford had been working on the newly discovered radio-activity of certain elements. He discovered that there were two types of radiation that were different from X-rays. He called them 'alpha' (α) and 'beta' (β) rays. Later he proved that 'alpha' rays consist of He-nuclei. Rutherford, in cooperation with Frederick Soddy, was the first one to prove Marie Curie's conjecture that radioactivity was an atomic phenomenon, which could lead to changes in the atom itself, from one element to another. This idea thus countered the prior idea that an atom was seen as the ultimate, indestructible form of matter: atoms could not change from one form (element) to another.

Figure 3.7: Marie Curie (1867-1934). From Wikimedia Commons, public domain.

Rutherford, in cooperation with Hans Geiger (one of the inventors of what we now call the Geiger counter) and Ernest Masden, built an apparatus that could count the α-particles. Moreover, he showed that the α-particles were He-nuclei with a positive charge of $2e$. In 1917, he showed that Nitrogen could become Oxygen by bombarding it with the α-particles. This was the first time that someone succeeded in artificially changing one element into another.

Scattering at a gold

As mentioned, Rutherford is responsible for overthrowing the plum pudding model and replacing it by our modern view: an atom is made of a tiny, positively charged nucleus with the electrons orbiting around it.

The start of this paradigm-shift was formed by Rutherford's observation that some of the α-particles were deflected by a thin metal sheet in front of his α-counter. This puzzled him as the plum pudding model could not explain this: when using that model the particles were either colliding or passing straight. Rutherford, Geiger and Marsden thus set up an experiment in which they led the α-particles scatter at a very thin gold foil to investigate further.

In the experiment, the source would emit α-particles through a small diaphragm onto the gold foil. The diaphragm made sure that all α-particles were traveling on the same line. After moving through the gold foil, the particles could be detected by looking via a microscope at the tiny light flashes an α-particle caused when hitting the detection screen. The microscope and detection screen could be placed under an angle with the original trajectory of the α-particles. By measuring at all possible angles, the scattering of the α-particles by the gold foil could be completely mapped and quantified.

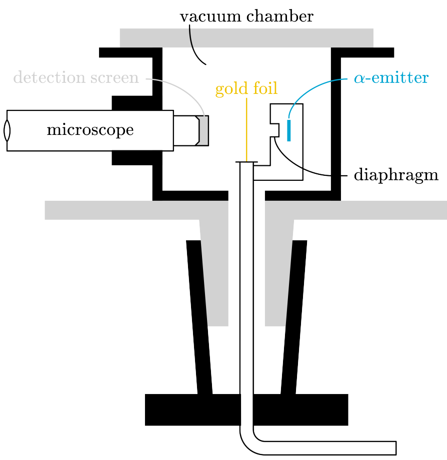


Figure 3.8: Experimental setup of α-scattering at a gold foil.

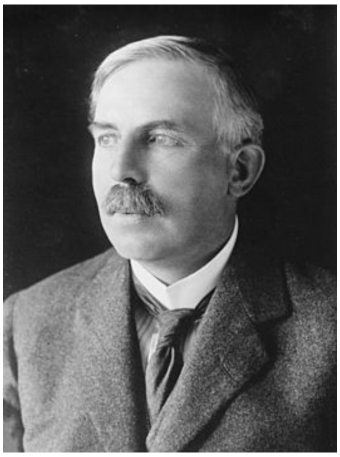


Figure 3.9: Ernest Rutherford (1871-1937). From Wikimedia Commons, public domain.

The story goes, that Rutherford's students would, together with Geiger, do the measurements as an assignment of their studies. The principle is simple: set the microscope under a known angle and, for a given period in time, count the number of hits. Repeat this for the next angle of the microscope. Obviously, the first measurements were all done on the side of the foil opposite to the α-emitter. One was expecting small deviations from the undisturbed trajectory.

When the experiments were basically done, so goes the story, still a student was left over that also needed an assignment. One of Rutherford's assistant suggested that this student would then measure with the microscope at the same side of the foil as the α-emitter. They did not expect anything to see, but they needed an assignment for this student. Whether the story of the student assignments is true or not, fact is that the team found also hits on the detector for angles of about 180°. That is, some α-particles seemed to bounce back from the foil!

There is no way that the plum pudding model could explain this. The argumentation to show that, goes as follows.

- The size of the atoms of gold is known: they are on the order of $r_0 \approx 10^{-10}$m.

This value can be found from the density of gold, the mass of a gold atom and the mass and volume of the gold foil (or any other amount of gold). By treating the atoms as small spheres that are stacked back to back, the size of the atom is easily found.

- An α-particle has a charge of $2e$ and is deflected by a gold atom due to the charge of the gold atom. As gold has number 79 in the periodic table, we know that the charge of a gold atom is +79e in the 'plum pudding' and −79e of all electrons floating in the pudding.

However, an α-particle is much heavier than an electron. Hence in the Coulomb interaction between the α -particle and an electron, the acceleration (of deflection) of the 'heavy' α-particle is negligible: the electrons are pushed out of the way (or actually attracted); they don't influence the trajectory of the α-particle.
It is the positive charge of the pudding itself, that does the deflection. The atom (i.e. the pudding) cannot move out of the way as it is part of the gold foil which is many orders of magnitude heavier than the incoming particle.

Rutherford knew the theory of Maxwell for electromagnetism and could estimate the force an α-particle would feel from the atom. He deduced that the force on the α-particle is always smaller than:

$$F_c \leq \frac{q_\alpha Q_g}{4\pi\varepsilon_0}\frac{1}{r_0^2} \tag{3.1}$$

with Q the charge of the atom (i.e. +79e), q_α the charge ($+2e$) of the α-particle, ε_0 the permittivity of vacuum ($\frac{1}{4\pi\varepsilon} = 9 \cdot 10^9 \quad \mathrm{Nm}^2/\mathrm{C}^2$) and r_0 the radius of a gold atom.

The deflection of the particle is biggest if the Coulomb force is perpendicular to the trajectory. So, we take that for our estimate. The true effect, when passing through the atom, will be smaller.

It is easiest to compute the change of momentum. The particle comes in with a know momentum p. If the change in momentum Δp is much smaller than p itself, the deflection will be small.

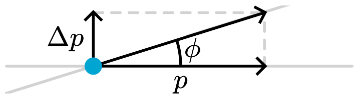


Figure 3.10: Relation of angle of deflection and change in momentum.

$$\tan\phi = \frac{\Delta p}{p} \Rightarrow \phi \approx \frac{\Delta p}{p} if \phi \ll 1 \tag{3.2}$$

The momentum change is due to the force F_c working for a time period Δt on the particle:

$$dP = Fdt - \rightarrow \Delta p \approx F_c \Delta t \tag{3.3}$$

The time the particle is in the atom is estimated as follows: the particle has a velocity v_0 and it has to travel a distance $2r_0$ through the atom, thus $\Delta t \approx \frac{2r_0}{v_0}$. We assume that the change in momentum is small, so that we can still use v_0 as an estimate of the velocity with which the α-particle travels.

If we put everything together, we find:

$$\frac{\Delta p}{p} \ll \frac{q_\alpha Q_g}{4\pi\varepsilon_0} \frac{1}{r_0^2} \cdot \frac{2r_0}{v_0} = \frac{q_\alpha Q_g}{2\pi\varepsilon_0} \frac{1}{r_0 v_0} \ll 1 \tag{3.4}$$

We have used the know charge of a gold atom ($79e$) and that of the α-particle, the radius of the gold atom and the incoming velocity of the α-particle, $v_0 \approx 1.6 \cdot 10^7 \mathrm{m/s}$.

With this estimate and the fact that Rutherford's gold foil was about 400 atoms thick, there is no way that we can explain α-particles bouncing back.

Rutherford and his colleagues, had no other option than to conclude that the positive charge of the gold atom must be confined to a much smaller part of space. After all, the only parameter in our estimate that is not measured is r_0. That was estimated based on the plum pudding model.

They redid the calculation, but now with r_0 as a free parameter to be backed out of the calculation. They changed the requirement of small scattering angles (i.e. small deviation from the original path) to the experimental finding that scattering angles of about 180° were possible. That gave that r_0 is on the order of 10^{-14}m.

Conclusion: the atom has a nucleus, which contains all of its positive charge and is much smaller than the atom itself. The electrons must orbit this nucleus as a mini-solar system. These electrons 'define' the size of the atom.

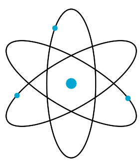


Figure 3.11: Rutherford's model of an atom.

This new model would spark a whole new set of questions, setting up one of the biggest changes in physics: Quantum Theory.

Collapse of matter? An immediate consequence of this new view on atoms and matter came from the analogy with Newton's work on the solar system and the Kepler Laws. In the case of the sun and planets, the interaction force is gravity: $\vec{F}_g = -G\frac{m_1 m_2}{r_{12}^2}\widehat{r_{12}}$. When dealing with a nucleus with its orbiting electrons the interaction force is the Coulomb force: $\vec{F}_C = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\widehat{r_{12}}$.

As both Gravity and Coulombs forces are central, conservative forces and are inversely proportional to the square of distance between the two interacting particles, the motion of a 'tiny' planet around the 'massive' sun is mathematically completely analogous to that of a 'tiny' electron around its 'massive' nucleus.

Thus an electron orbits the nucleus in an ellipse. Consequently, it is in a permanent state of acceleration. However, from Maxwell's theory of electromagnetism it is well known (already in the times of Rutherford as the theory of Maxwell dates back to around 1860) that accelerating charged particles radiate energy in the form of electromagnetic waves. This means that the electron constantly looses energy and thus moves to an elliptical orbit closer to the nucleus until, eventually, its orbit collapses onto the nucleus. This process would go very fast and matter in its present form could not exist. Now we know that the idea of an atom being a miniature solar system is wrong. But questions and dilemmas like these grew very quickly, giving rise to quantum mechanics and opening a whole new world: a completely different picture of things at small scales. A world with new rules and new consequences, where our intuition, based on daily life and large-scale structures composed of many, many atoms, fails.

Scattering Theory The work of Rutherford and co-workers forms the start of a new branch of physics: nuclear physics. By using radiation in the form of X-rays (i.e. high energy photons) and electrons or protons, physicists are able to probe the internal properties of molecules, atom, nuclei and even elementary particles (or at least, what we once thought were elementary particles).

The idea is to send high energy particles towards the object of investigation and look at the scattering that is a consequence of the interaction between the object and the incoming particles. The internal structure of the object dictates the scattering. Thus, by measuring the scattering features and back tracing the underlying physical interaction can be found.

It is done with facilities of a very large scale to research particles at the smallest scales. For instance, in CERN researchers accelerate particles (protons, electrons, etc) to velocities almost the speed of light. Then, they let these particles collide, that is undergo interactions involving enormous amounts of energy, and measure the fragments and all kind of exotic particles that result from these collisions.

Figure 3.12: Circular Accelerator of CERN depicted in its environment. ESO/José Francisco, licensed under CC-BY 4.0.

The principles used in scattering can be illustrated by revisiting Rutherford's experiment.

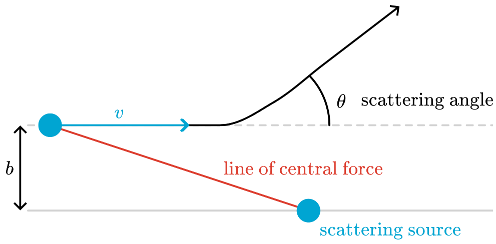


Figure 3.13: Scattering of an incoming particle at a fixed source.

Consider Figure 9: a particle of mass m and velocity v is moving towards a fixed second particle. The latter is fixed in the origin and act like a force-source. The force is central, i.e. works along the direction of the red line in Figure 9. In the drawing the forces is repelling and the incoming particle will deviate from its straight line. Eventually it will continue moving over a straight line, when the influence of the force is no longer felt. The angle of the new direction with the incoming one, is θ, the scattering angle. We are looking for the relation between b, the distance at which the incoming particle would have passed by the origin if there was no force and the scattering angle θ.

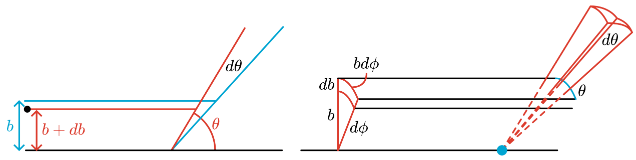


Figure 3.14: left: scattering in 2D, right: scattering in 3D.

In Figure 10 scattering in a 2D world and in the 3D world is schematically depicted. In the 3-dimensional world the scattering takes place in the solid angle $d\theta$. Like the 2d equivalent, where the scattering angle can go from 0 to 2π (that is the full circle), in 3d it goes from 0 to 4π reflecting that it is now a full sphere.

3.1.2 Kinetic theory of gases

Already in the 18^{th} century, work was done on what we call the kinetic theory of gases. The Swiss scientist Daniel Bernoulli proposed that gases were a large collection of molecules, i.e tiny particles moving in all directions. According to Bernoulli, their collision with walls was felt macroscopically as pressure and their averaged kinetic energy was in essence the temperature of the gas.

(a)

(b)

Figure 172: Two famous scientists working on the physics of gases.

It took a while before these ideas were accepted, partly because the law on conservation of energy was not fully developed. Moreover, people had difficulty accepting that at a molecular level collisions could be perfectly elastic.

With the further development of Thermodynamics, the kinetic theory of gases also refined. In 1856, August Krönig came up with a simple kinetic model for gases in which he only considered the possibility of translational motion of the molecules. In essence, he treated gas molecules as point particles. A year later, Rudolf Clausius incorporated the possibility of rotation and vibrations. Two years after this, James Clerk Maxwell continued along this line. He found the velocity distribution of the molecules and established a firm connection between temperature and the average kinetic energy of a molecule. However, he also noted that the theoretical predictions were not in line with experiments. What was the problem?

3.1.2.1 Specific heat of gases

For ideal gases, we have the ideal gas law: $pV = nRT$ with n the number of moles of the gas in question. Or written in terms of number of molecules, N, it reads as: $pV = NkT$, k being the Boltzmann constant.

The ideal gas law helps in understanding how gases behave under changing conditions. For instance, if we compress a given amount of gas, we may expect that the pressure goes up. But we also see that this depends on whether or not the temperature changes. And in principle the temperature will change.

If we would do a compression experiment with a fixed number of molecules, N, and we would compress the gas such that no heat can escape from the container, then the changes in temperature, volume and pressure are such that $pV^\gamma = const$. This is called adiabatic compression. The quantity γ is the ratio of the specific heat at constant pressure over the specific heat at constant volume. Both these quantities are easily measured in experiments and, hence, γ can be found for many gases.

The kinetic theory predicts γ for various classes of gases. For instance, for monatomic gasses as Helium, it is $5/3 \approx 1.667$; for diatomic gases, such as Oxygen or Hydrogen, it should be $9/7 \approx 1.286$. And so on. Moreover, γ does, according to the kinetic theory, not depend on temperature.

In the table below, the ratio of the specific heats c_p/c_V is listed for a number of gasses.

Gas	c_p/c_v	kin.gas.th.
He	1.663	1.667
Ne	1.667	1.667
Kr	1.656	1.667
Br_2	1.28	1.286
Cl_2	1.34	1.286
H_2	**1.405**	1.286
N_2	**1.40**	1.286
O_2	**1.395**	1.286

As we see, for the noble gases it is quite ok (at $T = 293K$!), but not so for the diatomic gases.

For really high temperatures (~2000K) for both O_2 and H_2, γ it is close to 1.286. But if we go to low temperature, their respective γ's increase and H_2 reaches a value of 1.66! Hence, Maxwell concluded, that the laws of classical mechanics could not be the final answer.

As we have seen when discussing Rutherford's experiment, in the early twentieth century more cracks became visible. These led scientists to Quantum Mechanics.

3.1.3 The problem with Maxwell's equations

In the early 1860s Maxwell extended Ampères' law, in combination with Gauss and Faraday laws. This led to four coupled differential equations describing the generation of electromagnetic fields from charges and currents. They are now just known as *the Maxwell equations*. They read in modern (differential) notation as follows for the electric $\vec{E}(\vec{x}, t)$ and magnetic $\vec{B}(\vec{x}, t)$ field in free space

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} &= \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \end{aligned} \tag{3.5}$$

With $\rho(\vec{x})$ the charge density distribution and $\vec{J}(\vec{x}, t)$ the electric current density, and constants ε_0 the vacuum permittivity and μ_0 the vacuum magnetic permeability.

You will learn all about Maxwell's equations in classes on *electromagnetism*. The mathematics of these equations is quite difficult as each equation is $3D + t$ and the equations are coupled.

In vacuum ($\rho = 0$ and $\vec{J} = 0$) we can simplify these equation. Furthermore, we could look at 1-dimensional cases, that is the electric field has only a component E_y which is only depending on time t and the x-coordinate. This leads us to the *wave equation*

$$\frac{\partial^2 E_y}{\partial x^2} - \varepsilon_0 \mu_0 \frac{\partial^2 E_y}{\partial t^2} = 0 \tag{3.6}$$

This equation describes the propagation of the electric field through vacuum (you can of course derive the same equation for the magnetic field). In the wave equation a second derivative in space is coupled to a second derivative in time. The solution to this differential equation is $E_y(x, t) \propto \cos(kx - \omega t)$, with the *wave number* k related to the wave length $k = 2\pi/\lambda$ and the angular frequency ω to the frequency ν according to $\omega = 2\pi\nu$. Like for all waves, the frequency, wave length and velocity of the wave are coupled: $\nu\lambda = c$ with c the speed of the wave, i.e. in this case the speed of light.

Light is identified as an electromagnetic wave and from the wave equation we see that the wave velocity is given by

$$\frac{1}{\sqrt{\varepsilon_0 \mu_0}} \equiv c = 2.998 \cdot 10^8 \quad m/s \approx 300,000 \quad km/s \tag{3.7}$$

If the Maxwell equations are laws of physics all inertial observers must be able to write down the equation in the same form. Therefore for an observer S', traveling at constant velocity $V\hat{x}$ with respect to S, we would write down the wave equation for a field that propagates only along the x-direction with amplitude in the z-direction (without loss of generality) $\vec{E} = \left(0, E_y(x, t), 0\right)$ as

$$\frac{\partial^2 {E'}_{y'}}{\partial x'^2} - \varepsilon_0 \mu_0 \frac{\partial^2 {E'}_{y'}}{\partial t'^2} = 0 \ (*) \tag{3.8}$$

This has exactly the same form as for S which is good if it should represent a physical law. However, for S' the speed of the wave is also given by $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$. As the speed is covered by universal constants ε_0, μ_0, the speed is the same of S and S' or in other words $c = c'$! This is not what should happen! From the Galilean Transformation we know that we should find the same form, but with $c' = c - V$ the relative velocity of the two observers.

If we apply the coordinate transformation from $S \to S'$ according to the Galilean Transformation, the wave equation reads thus as

$$\frac{\partial^2 E_z}{\partial x'^2} - \frac{1}{c^2} \frac{\partial^2 E_z}{\partial t'^2} - \frac{V^2}{c^2} \frac{\partial^2 E_z}{\partial x'^2} + \frac{2V}{c^2} \frac{\partial^2 E_z}{\partial x' \partial t'} = 0 \tag{3.9}$$

Now we still need to find a transformation $E_z \to {E'}_{z'}$ (and $c' \to c$) trying to retrieve the general form of the wave equation, but there is no such transformation. Therefore the wave equation of electromagnetic waves is not Galilei invariant at all! This was a serious issue at the time.

3.1.3.1 Hypothesis of the aether

As light is a wave, people naturally thought there must be a medium to transport the wave, *something* must be oscillating. Vacuum was considered nothing, not something. A water wave, needs water as medium to transport the wave; the water oscillates. Or take sound waves, they need gas, liquid or a solid to oscillate. What could be the medium that light, electromagnetic waves, use to oscillate? This medium must be all around us, in the space between the sun and earth, just everywhere. To save the Galilei invariance of Maxwell's equations this also needs to be a very special kind of medium that behaves differently than other media. This strange hypothetical medium was termed *aether* (ether). The search for the properties of the aether lead to the Michelson-Morley experiment - which showed that there was no aether at all! Lorentz and Fitzgerald found an ad hoc way to save the day by postulating an adapted version of the Galilean Transformation and a contraction of length. Later more about that, and how Einstein showed that all of this ad hoc business is not needed, things follow directly from his second axiom.

3.1.4 The Michelson-Morley experiment

The Michelson-Morley experiment was performed in between 1880-1890 to investigate properties of the hypothetical aether. The experiment returned a null-result, i.e. there was no sign of the existence of the aether - and to this day there is none.

The idea is to check the speed of light for two observers S and S'. One is moving with respect to the other with the highest possible speed, the orbit speed of the earth around the sun $\sim$ 30 km/s. Of course, that is still only 10^{-4} compared to 300,000 km/s of the speed of light but the effects could be measured spectroscopically by interference of light.

The experiment essentially consists of a Michelson interferometer. Light is send to a 50/50 beam splitter such that half of the light is reflected up towards arm L_1 and half is transmitted to arm L_2. The mirrors at the end of each arm reflect the light back. On the way back again half of the light is transmitted and reflected at the beamsplitter, such that half of the light from both arms is now traveling downwards towards the image plane/camera. At the image plane the light from both arms forms an interference pattern, depending on the path length difference induced by the difference of $L_1 - L_2$.

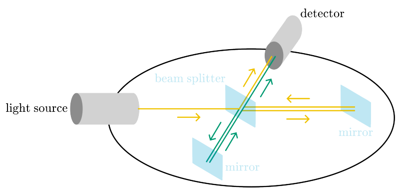
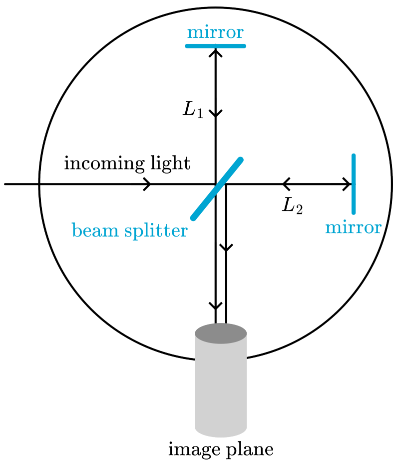


Figure 3.18: Michelson & Morley setup



The whole setup is mounted for stability on a heavy table that is floating in liquid mercury, to reduce vibrations coupling to the setup. If now one arm is parallel to the earth's orbit with $V = 30\text{km/s}$, while the other is perpendicular to it, there will be some difference between the length of the two paths traveled: $\Delta\lambda_1$. If we rotate the setup by 90 degree (easily done in the mercury bath), then the roles of L_1 and L_2 are exchanged, leading to another phase shift $\Delta\lambda_2$. Therefore after rotation the fringes of the interference pattern on the detector should shift as

$$\Delta\phi = 2\pi\frac{\Delta\lambda_1 - \Delta\lambda_2}{\lambda} = 2\pi\frac{(L_1 + L_2)}{\lambda}\frac{V^2}{c^2} \tag{3.10}$$

If we fill in the numbers $\lambda = 550$ nm, $L_1 + L_2 = 11$ m and $V^2/c^2 = 10^{-8}$ this results in an expected $\Delta\phi = 0.4\pi$. However, Michelson and Morely found only $\Delta\phi \leq 0.01\pi$. The experiment to find the aether failed.

Physics was in serious trouble until 1905.

NB: Back in the days, white light was used for the actual measurement and monochromatic coherent light of e.g. a sodium lamp for alignment. As white light produces a colored interference pattern which is much easier to observe visually. Otherwise temperature changes or vibrations, resulted in constant fringe drift. Today monochromatic laser light can be used in combination with environmental temperature control better than to 0.1 C and sensitive CCD cameras. Today experiments have confirmed the null-result of Michelson and Morley but to much better precision. The anisotropy in the speed of light is $c_\perp/c_{||} \leq 10^{-17}$.

Although the proposed hypothetical medium aether does not exist, as proven a long time ago, the terminology did not drop from everyday language.

3.1.5 Einstein's axioms

In 1905 Einstein formulated his special theory of relativity with the article *Zur Elektrodynamik bewegter Körper*, Annalen der Physik, **17**:891-921, 1905. He choose the Maxwell equations and the Michelson Morely experiment as a starting point for this argument to arrive at

1. The laws of physics are the same in all inertial frames of reference.
2. The velocity of light in vacuum is the same in all inertial frames.

That does not sound like a lot or world changing, but it certainly was. You can directly see that the second axioms violates Galilean addition of velocities, but that is what was found experimentally by Michelson and Morley!

If you think these two axioms stubbornly through and take their consequences seriously, things get confusing, surprising and almost impossible to believe. Nevertheless, we will do this. Why? Because nature is this way, whether we like it or not.

Extra reading with a historic perceptive. In a 200 page book Wolfgang Pauli -*Theory of Relativity*, Dover (the original German version is available online *Relativitätstheorie*) - summarizes all that was known about special relativity as a request made by this PhD advisor Arnold Sommerfeld. It is worth a read, although the notation is a bit outdated.

Extra reading Hoe de ether verdween uit de natuurkunde. This article by Jos Engelen in the *Nederandse Tijdschrijft voor Natuurkunde* explains the Michelsen-Morley experiment, places it into historic perspective and then adds the work of Lorentz, Poincaré and Einstein leading to the Lorentz transformation.

Exercise 3.20:

Assume the Michelson-Morley experiment uses arms of length $L = 11 \quad \mathrm{m}$ and an aether wind speed v due to the motion of the earth around the sun.
Distance earth-sun: $150 \cdot 10^6$ km.

1. Calculate the expected time difference Δt between light traveling parallel and perpendicular to the aether wind.

The sun itself is orbiting the center of our Milky Way at an even higher speed: 250km/s.

2. What would this mean for the expected time difference in the Michelson and Morley experiment?

Note: in the experiment of 1887, Michelson and Morley had used multiple mirrors in their set up for each of the two light beam paths to make the traveling length of the light, from the splitter to the mirror and the edge of the table and back, much longer than only the radius of the table and back. This is how they achieved a path length of 11 m. The table itself was of course not of a diameter of 22 m.

Solution 3.21: Solution to Exercise 1

The orbit of the earth around the sun is almost circular. Thus, we can estimate the velocity of the earth as $V = \frac{2\pi R}{T}$ with $R = 150 \cdot 10^6$ km and $T = 1 \quad \mathrm{year} = 31.610^6$ s. This gives $V = 30$ km/s.

We compute the traveling time from light leaving the beam splitter, reflecting at the mirror on the side of the table and reaching the beam splitter again. The rest of the path is identical for both light beams and does not lead to a time difference.

Time for light parallel to V:

- one part - tail wind from aether and velocity (according to Classical Mechanics with Galilean Transformation) $c + V$.
- Other part: head wind with velocity $c - V$. Thus traveling time:

$$t_{//} = \frac{L}{c-V} + \frac{L}{c+V} = \frac{2L}{c}\frac{1}{1-\frac{V^2}{c^2}} \tag{3.11}$$

Time to travel perpendicular to V:

$$t_{\perp} = \frac{L}{\sqrt{c^2-V^2}} + \frac{L}{\sqrt{c^2-V^2}} = \frac{2L}{c}\frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \tag{3.12}$$

Putting in the numbers, we find $\Delta t = 3.67 \cdot 10^{-16}$s

This time difference may be way to small to measure. And indeed, no 'stop-watch' experiment will work. But Michelson & Morley used interferometry, i.e. interference of light. So, relevant is the difference in phase of the two light beams. This can be assessed by turning the time difference into a length: $\Delta s = c\Delta t = 1.1 \cdot 10^{-7}$m. Compare this to the wave length of the (yellow) light used by Michelson and Morley: $\lambda \approx 500 \quad \mathrm{nm} = 5 \cdot 10^{-7}$m. Conclusion: the expected time difference is well in reach of interferometry.

Exercise 3.22: Deterministic nature of physics 🌶️

Write down whether your deterministic view on nature (everything in nature can be described by physics) has changed given the three-body problem.

3.1.6 Exercises, examples & solutions

Updated: 04 feb 2026

3.2 Special Relativity - Lorentz Transformation

Updated: 04 feb 2026 As we discussed, in the second half of the nineteenth century it became clear that there was something wrong in classical mechanics. However, people would not easily give up the ideas of classical mechanics. We saw that the luminiferous aether was introduced as a cure and as a medium in which electromagnetic waves could travel and oscillate. Moreover, Lorentz and Fitzgerald managed to find a coordinate transformation that made the wave equation of Maxwell invariant. Fitzgerald came even up with length contraction: if the arm moving parallel to the aether of the interferometer of Michelson and Morley would contract according to $L_n = L\sqrt{1 - \frac{V^2}{c^2}}$ then, the M&M experiment should result in no time difference for the two paths, in agreement with the experimental findings. However, there was no fundamental reasoning, no physics underpinning the transformation and the length contraction. It worked, but had an ad hoc character. Very unsatisfying for physicists!

And as we have mentioned, it took the work of a single man to change this and underpin the Lorentz Transformation, making Classical Mechanics a valid limit of Relativity Theory, only applicable at velocities small compared to the speed of light and to small distances compared to those of interest in cosmology.

Figure 3.23: Albert Einstein (1879-1955). Picture from Wikimedia Commons, public domain.

Lorentz Transformation

$$\begin{aligned} ct' &= \gamma\left(ct - \frac{V}{c}x\right) \\ x' &= \gamma\left(x - \frac{V}{c}ct\right) \\ y' &= y \\ z' &= z \end{aligned} \tag{3.13}$$

with

$$\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{3.14}$$

But there is more! Einstein also changed our view on the universe and on time itself. In the world of Newton and Galilei, people could not even think about relativity of time. Of course time was the same for everyone. There was only one time, one master clock - the same for all of us. It is hard coded in the Galilean Transformation:

Galilean Transformation

$$\begin{aligned} t' &= t \\ x' &= x - Vt \\ y' &= y \\ z' &= z \end{aligned} \tag{3.15}$$

Now, if we compare GT to LT, we see that with the Lorentz Transformation this is no longer true: different observers may have different time. We will see that this has very peculiar consequences, some of which are very counterintuitive. However, they have been tested over and over again. And so far: they firmly hold. And there is no other way than to accept that the world and our universe is different from what we thought and from what we experience in our daily lives.

Do note, that the Galilei Transform is a limit of the Lorenz Transformation. If we let $c \to \infty$, we see that $\gamma \to 1$ and $\frac{V}{c} \to 0$. And this gives us: $t' = t$ and $x' = x - Vt$, that is the Galilean Transformation! Now, this should not come as a surprise (even if it for a moment did). After all, Classical Mechanics does an outstanding job in many, many physics problems and the agreement with experiments is excellent.

3.2.1 The Lorentz Transformation

The way we wrote down the Lorentz transformation is a bit particular in a sense that we combine time t with the speed of light c into the "time" axis ct which now has unit length. We can do this as c is constant for all observers independent of their frame of reference. The speed of light is said to be a **Lorentz invariant**. In this notation the transform between S and S' (moving with velocity V away) is *easy to remember*!

3.2.1.1 S and S'

We will discuss most of the consequences for two observers S and S', traveling with a constant velocity $\vec{V}$ with respect to each other. They have taken their x, resp. x' axis parallel to $\vec{V}$. Hence, we only need to talk about V, knowing that this is the only component of the relative velocity between the two observers and that it is along the x, x' axis.

Furthermore, their y and y' coordinates are taken in the same direction. This also holds for the z-component. Finally, when S and S' pass each other (they are then both at the same point), they put their clocks to zero: $t = 0$ and $t' = 0$.

Note: S is sitting in her origin $\mathcal{O}$ (with coordinates, according to S $(x, y, z) = (0, 0, 0)$) and stays there. Similarly for S' who is sitting in $\mathcal{O}'$ (with coordinates, according to S' $(x', y', z') = (0, 0, 0)$).

The standard sketch is given in the figure below.

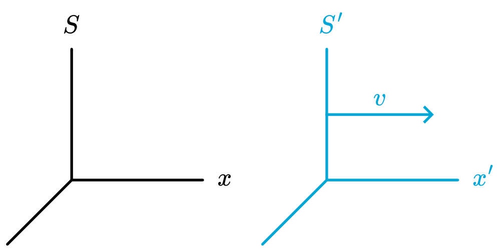


Figure 3.24: S and S': relative velocity parallel to the x and x' axes.

N.B. It is crucial to be very precise in your notation when it comes to coordinates and quantities. For instance: S might talk about the x-component of the velocity of an object and denote this by v_x. S', on the other hand can also talk about that component, but will not call it the x-component: in the world of S' x "does not exist", only x' does. So it is better to write $v'_{x'}$ for the x'-component of the velocity of the object according to S'. It may look cumbersome, and to a certain extent it is, but it actually does make sense. S' would say that this component is $\frac{dx'}{dt'}$ both space and time having a prime. Hence, naturally S' would talk about $\vec{r}' = x'\hat{x}' + y'\hat{y}' + z'\hat{z}'$ or $\vec{v}' = v'_{x'}\hat{x}' + v'_{y'}\hat{y}' + v'_{z'}\hat{z}'$

3.2.1.2 Lorentz Transformation and its inverse

The Lorentz Transformation, like the Galilean Transformation is a communication protocol for S and S'. It allows them to interpret information that they get from each other in their own 'world', i.e. coordinate system.

For instance, if S sees an object moving with v_x, S' can 'translate' this information via the Lorentz Transformation into $v'_{x'}$ and $v'_{y'}$ or so if applicable. Of course, S also needs such a translation scheme when receiving information from S'. That is: S needs the inverse transformation.

Luckily, the inverse is very easy to reconstruct from the Lorentz Transformation itself. LT from S to S' is

$$\begin{array}{rcl} ct' &=& \gamma\left(ct - \frac{V}{c}x\right) \\ x' &=& \gamma\left(x - \frac{V}{c}ct\right) \\ y' &=& y \\ z' &=& z \end{array} \tag{3.16}$$

The inverse is found by invoking 'relativity', after all it is called Relativity Theory. If S sees $'$ moving at a constant velocity V, then - because motion is relative- S' will say that S moves with $-V$. And thus, if S' writes down the Lorentz Transformation, she uses $-V$.

The inverse is therefore given by

$$\begin{array}{rcl} ct &=& \gamma\left(ct' + \frac{V}{c}x'\right) \\ x &=& \gamma\left(x' + \frac{V}{c}ct'\right) \\ y &=& y' \\ z &=& z' \end{array} \tag{3.17}$$

with the *Lorentz factor* $\gamma(V) \equiv \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \geq 1$. Note that as γ is quadratic in V, both S and S' use the same value! That is why we don't talk about γ': it is equal to γ.

The structure of the formulas is very symmetric and therefore needs little remembering.

From the Lorentz transformation it is clear that time is not universal anymore ($ct' \neq ct$ in general). This is a large step from Newton and Galileo. Now the time coordinate is mixed somehow with the space coordinate depending on the speed V.

3.2.1.3 Lorentz factor

The Lorentz factor (or γ-factor)

$$\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \geq 1 \tag{3.18}$$

is a dimensionless constant depending on the ratio of the velocity V to the speed of light c. Sometimes this ratio V/c is abbreviated further as $\beta \equiv \frac{V}{c} \leq 1$. For the ratio we know that it is smaller than 1 as c is a limit velocity. From that it follows that the γ-factor is always equal to or larger than one, $\gamma \geq 1$.

In many exercises the speed V is given already as fraction of c, e.g. $V = 0.8c$. Analytically only for very few speeds a nice γ-factor is computed. These are for instance

$$\begin{array}{rcl} V = \frac{3}{5}c & \Leftrightarrow & \gamma = \frac{5}{4} \\ V = \frac{4}{5}c & \Leftrightarrow & \gamma = \frac{5}{3} \\ V = \frac{12}{13}c & \Leftrightarrow & \gamma = \frac{13}{5} \end{array} \tag{3.19}$$

Note that this list goes on for ever: there is a simple rule to find the triplets. Think about it yourself. Hint: the first one uses $(3, 4, 5)$, the third one $(5, 12, 13)$. What is special about them?
$5^2 - 4^2 = 5 + 4 = 3^2$ and $13^2 - 12^2 = 13 + 12 = 5^2$.
Do you see the pattern? Can you derive the general rule? What is the next one? How about $(7, 24, 25)$?

3.2.1.4 In the limit

In the **limit of low speeds** with respect to the speed of light $\frac{V}{c} \ll 1 \Rightarrow \gamma = 1$. Practically, this happens for about $V < 0.1c \sim 30.000\text{km/s}$. In this limit the Lorentz transformation also reduces to the Galilean Transformation.

$$\begin{array}{rcl} ct' & = & ct \\ x' & = & x - Vt \\ y' & = & y \\ z' & = & z \end{array} \tag{3.20}$$

In the **limit of infinity speed of light** ($c \to \infty$) the γ-factor is again one: $\gamma = 1$ and the ratio $V/c \to 0$. Also here the LT reduces to the GT. The case of infinite speed of light represents the case that GT is generally valid, i.e. $c' = c + V$.

It is always important to verify that an extension of a well-established theory reproduces the results of that theory in the domain where it has been experimentally validated.

Historical context

Lorentz did not derive the transformation that now has his name, based on Einstein's axioms. Lorentz, however, saw that Maxwell's equations were not GT invariant, therefore he tried to find a transformation under which they were invariant. He did so (with a bit of help from Poincaré afterwards). FitzGerald did also derive the transformation, but too did not understand its implications.Hendrik Lorentz (1853-1928). From Wikimedia Commons, public domainGeorge Fitzgerald (1851-1901). From Wikimedia Commons, public domainBefore Einstein's idea spread, Lorentz thought about the transformation as a fix to Galilean Transformation. Later he understood, of course. Unfortunately, Fitzgerald did not live long enough to see the first publication of Einstein on Relativity in 1905.The electro-magnetic wave equation can be transformed from to . And indeed, if you would do that, you would find that the wave equation maintains its form with the same , not a new . Lorentz had found this, but it was Einstein who underpinned and generalized the use of the Lorentz Transformation to all mechanics, replacing the Galilean Transformation.

Exercise 3.25:

Close your book, laptop. Shut down your screen, put aside your mobile, tablet. Put away you notes and put an empty clean sheet in front of you. All you have is that sheet of paper, one pen and your brain.

- Write down the Lorentz Transformation and its inverse.
- Repeat so you don't forget it (for the rest of your life: Every physicist ought to know the LT by heart ;-)).

3.2.2 Length contraction & Time dilation

3.2.2.1 First Implications

As we have seen, we need to use the Lorentz Transformation instead of the Galilei one when two observers, S and S', want to exchange information. What changes if we do so? Let's first do some examples and see some of the consequences and the 'strange' conclusions we need to draw.

Note: we will frequently use high velocities and large distances. It is convenient not to write these in units like m and m/s. The numbers in front of them become so large that keeping an overview becomes cumbersome. Therefore, we will change to a different unit for distance: the light second. That is per definition the distance a photon of light ray travels in one second:

$$1 lightsecond = 1 \quad \text{ls} = c \cdot 1 \quad \text{sec} = 3.0 \cdot 10^8 \quad \text{m} \tag{3.21}$$

For instance, it takes a photon about 8.3 minutes to travel from the sun to the Earth. Thus, the distance from the sun to the Earth is 8.3 lmin $= 498$ ls. That is equivalent to $150 \cdot 10^6$ km.

Example: spaceship

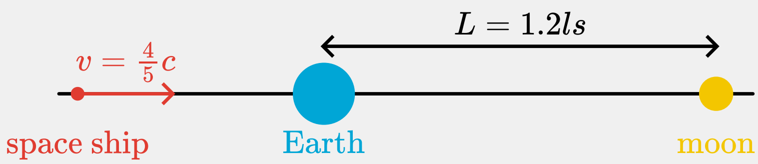
A spaceship is flying at a velocity $0.8c$ past Earth in the direction of the moon. The moon is at a distance of 1.2 ls (that is some $3.6 \cdot 10^8$m) from the Earth. The clocks on Earth and in the spaceship are set to zero when the spaceship passes the Earth.

At time $ct = 1.7$ ls observer S of the Earth observes that a big comet strikes the moon surface.

When does S', who is on the spaceship, see this happening?

Solution

First we make a sketch.



Next, we need to carefully clarify what we mean by observe, know, see. This is very important as observations are made by someone at a certain time, while being at a certain position. Since now both time and place information gets into the transformation, being sloppy might lead to very strange and wrong conclusions.

Thus, we will from now on, specify **Events**. An event is a physical phenomenon happening at a certain place at a certain time. For instance, you catching a frisbee at 12:45 (i.e. t_f) on the campus (at location x_f, y_f, z_f). This will be denoted as:

$$frisbeecaught : E_f = \left(ct_f, x_f, y_f, z_f\right) = (.., .., .., ..) \tag{3.22}$$

That is, four coordinates are specified (in m or ls or ...). Note: this is information as used by S: the coordinates do not carry a prime.

So, back to our example: we have our first event:

$$S observes `comethitsmoon' E_1 = (ct_1, x_1, y_1, z_1) = (1.7, 0, 0, 0) \tag{3.23}$$

What does this mean? Observer S, who is sitting in $\mathcal{O} = (0, 0, 0)$ literally sees that the comet hits the moon. He does so at $ct_1 = 1.7$ ls. In terms of physics: a photon hits his eye at ct_1. The observer has zero-size, that is everything he observes is done at $(0, 0, 0)$.

Now, we need to realize, that the actual impact of the comet took place earlier. By how much? Well, a photon that was generated at this moment of impact due to the impact will have to travel 1.2 s to reach S. That requires 1.2ls, as photons travel with the speed of light.

Thus, S concludes that the actual impact -which is event E_2- took place at $ct_2 = 0.5ls$ and he writes down:

$$comethitsmoon E_2 = (ct_2, x_2, y_2, z_2) = (0.5, 1.2, 0, 0) \tag{3.24}$$

Again notice that we have updated this event not only by using the actual time, but also the actual place, i.e. at x_2.

S passes this information on to S'. She has to translate it to her own coordinates and uses for that the Lorentz Transformation.

First, she needs to calculate the γ-factor:

$$\gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{16}{25}}} = \frac{5}{3} \tag{3.25}$$

Now she computes her coordinates for the same event:

$$\begin{aligned}
ct'_2 &= \gamma\left(ct_2 - \frac{V}{c}x_2\right) = \frac{5}{3}\left(0.5 - \frac{4}{5}1.2\right) = -0.767 \text{ ls} \\
x'_2 &= \gamma\left(x_2 - \frac{V}{c}ct_2\right) = \frac{5}{3}\left(1.2 - \frac{4}{5}0.5\right) = 1.333 \text{ ls} \\
y'_2 &= y_2 = 0 \\
z'_2 &= z_2 = 0
\end{aligned} \tag{3.26}$$

We will not further deal with the y and z coordinates as they are trivial.

But, we might get our first surprise here. According to S the impact of the comet happens at $t = 0.5$ s. That is at a positive time. Then, the spaceship has passed the Earth and is on its way to the moon. Actually, at $t = 0.5$s the location of spaceship is, according to S: $x_{SS}(t) = Vt = \frac{V}{c}ct \to x_{SS}(0.5 \text{ s}) = \frac{4}{5}0.5 = 0.4$ ls. spaceship is already at 1/3 of the distance to the moon.

So far nothing strange.

But now we consider S'. She says: the impact of the comet was at $t' = -0.767$. This means that according to her, the impact took place when she was still approaching the Earth. After all, negative times mean that spaceship is approaching the Earth (and is to the left of it in our sketch), while positive times mean that spaceship has passed the Earth and is moving away - thus is at the right side of Earth in our sketch.

And this is so according to both S and S'. They may use different times, but they have set their clocks to zero when Earth and spaceship were in 'the same position'.

Ok, let's be puzzled for a while: how can S' at the same time be both at the left side and at the right side of the Earth? That doesn't make any sense!!!! What is **wrong** with this new theory? The answer is: **nothing!**

It is us, mixing stuff up. Who said that it is 'at the same time'?!? Nobody (with perhaps for a moment us as the exception). S and S' agree upon the event: a comet hits the moon. This physical phenomenon is not disputed at all. It happened. They don't agree that it took place at the same time according to their clocks.

But this is not all: according to S at the moment of the impact spaceship was at a distance of $1.2 - 0.4 = 0.8$ ls from the moon. But S' just calculated that she was 1.33 ls from the moon. One cannot be at two different distance form the moon at the same time!

Ok, let's push this somewhat further and see if we can get a contradiction.

We do know, from S that the event took place at $ct_2 = 0.5$ ls. Then, definitely S' has passed Earth. S has reconstructed this event from observation Event E_1 S' got the information of event E_2 from S and backed out the coordinates of the event in her coordinate system. From these data, S' can easily predict when she will see the impact. That is obviously later than the time of the event: the photons have to travel to her. How can we compute when S' literally sees the event?

That is remarkably easy: we know that according to S' the event tokes place at $(ct'_2, x_2') = (-0.767 \text{ ls}, 1.333 \text{ ls})$. At that moment and that place a photon was generated that moves in her direction. Since the velocity of each photon is always c, we can easily find the time when S' sees the photon, i.e. detect it at location $x' = 0$. The photon has to travel a distance 1.33 ls at a speed of $1c$. That will take 1.33 s. The photon started traveling at time $ct_2 = -0.767$. Its trajectory according to S' is $x'_p(t') = x'_p(0) - c(t' - t'_2)$.

Thus, the photon gets measure at event E_3: $x_3{}' = 0 \rightarrow ct'_3 = x'_2 + ct'_2 = 0.567$ ls. Thus we have our third event:

$$spaceshipobservesimpactingcomet : E_3 = (ct'_3, x'_3) = (0.567, 0) \tag{3.27}$$

And as we by now kind of expected: indeed, then is spaceship on the right side of the Earth. What does S say about this event? He receives the coordinates of $E3$ from S' and plugs them in, in the inverse LT:

$$\begin{aligned} ct_3 &= \gamma\left(ct'_3 + \frac{V}{c}x'_3\right) = \frac{5}{3}\left(0.567 + \frac{4}{5}0\right) = 0.945 \text{ ls} \\ x_3 &= \gamma\left(x'_3 + \frac{V}{c}ct'_3\right) = \frac{5}{3}\left(0 + \frac{4}{5}0.567\right) = 0.756 \text{ ls} \end{aligned} \tag{3.28}$$

Now does this make any sense? It does! If we concentrate on S only and what he observes and knows:

- E_1 - S observes -comet hits moon: $(ct_1, x_1) = (1.2, 0)$ls
- E_2 - the comet actually hits the moon: $(ct_2, x_2) = (0.5, 1.2)$ls
- E_3 - S' observes that the comet hits the moon: $(ct_3, x_3) = (0.945, 0.756)$ls

Obviously, if the actual impact is at positive t, then S' will see it before S does as for positive time t S' is closer the moon than S. And this is all reflected in the events. Moreover, if you would compute the events as S will model things, you will find event E_3 just based on event E_2 and the motion of spaceship according to S (and when it will encounter a photon that was generated at the actual impact of the comet on the moon). Do the calculation yourself and see, that nothing strange happens.

We can draw the position of Earth, moon and spaceship in space-time plot. It is customary to use as horizontal axis the x or x' coordinate and as the vertical one ct or ct'. S will see the Earth and moon standing still and thus draw a vertical line in the space-time diagram for each of them: they do not change position, but their time is changing, i.e. the clock ticks. S would draw for spaceship a straight line moving from left bottom to upper right as the spaceship moves in the positive direction.

Similarly, S' will draw a vertical line for spaceship itself, as in the frame of reference of S' the spaceship, obviously, does not move. The Earth and moon move to the left, thus their trajectories are straight line from the bottom right to the upper left in the (x', ct')-diagram.

At some moment in time-space the comet impacts the moon and a photon is moving in the negative x-direction towards the Earth. Somewhat later, this photon is received by Earth. In the (x, ct)-diagram this is a straight line from lower right to upper left.

In the animation below the whole scenery is shown from the perspective of S on the left side and from S' on the right side. The diagrams are made such, that the event "spaceship passes Earth" is simultaneous in both diagrams, i.e. it happens for both observers at their time equal to 0. All other events happen at different times according to the clocks of the observers.

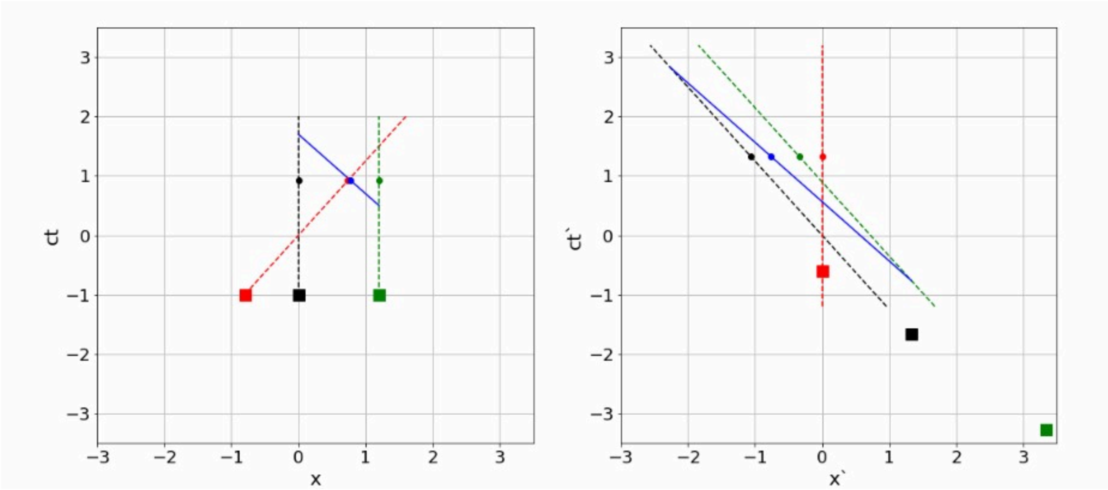


Figure 3.27: An animation of events as seen by different observers.

An animation is given above.

- the three squares represent the position of Earth, moon and spaceship according to S at $ct = -1$ls. In the diagram for S, these three are, of course, on a horizontal line as they are at the same time according to S. However, S' sees that differently: there are absolutely not at the same time!!!
- Earth, moon and spaceship do travel in the space-time diagrams. Their trajectories are shown by dashed lines. Their space-time location is represented by the (moving) dots. The diagrams are made such, that indeed both observers pass each other at $ct = ct' = 0$ and $x = x' = 0$. The dots represent, where according to S (left diagram) and S' (right diagram) Earth, moon and spaceship are at a certain time on the clock of that observer. Note that both position and time have really different values if you compare the diagrams of S and S'.
- In both diagrams, at some point in time the comet impacts the moon and a photon starts traveling in the negative x and x'-direction. The photon is shown by the blue dot. Again nothing happens at the same time. But the order of events is the same: first the photon is emitted and only after that it is observed. That should of course hold!
- Notice that the photon is emitted at $ct = 0.5$ls according to S and observed at $ct = 1.7$ls. So for S, the photon traveled for 1.2ls (and covered a distance of 1.2ls: of course, photons travel with velocity c). However, for S' this is quite different: the photon is emitted at $ct' = -23/30$ls, that is much earlier than S reports. Moreover, it is only registered by S on $ct' = 85/30$ls. It traveled for 3.6 seconds on the clock of S'!!

Puzzled by this all? Confused? Hard to believe?

Welcome the 'Magical World of Relativity'. And don't worry: you will get used to it. Moreover, we will develop a mathematical framework that helps us and prevents our failing intuition to take the wrong path.

Conclusions:

- We need to be careful with interpreting distances and times, things are not what they seem at first glance.

- Within the framework of one observer nothing funny happens.
- We better work with well defined events: they represent physical phenomena happening. Both observers will agree upon these and on the logic, e.g. first the impact than the observation of a photon - not the other way around!

3.2.2.2 Time & Space

Here we have a look at the consequences of axioms 1 & 2. We know how two observers S and S' (moving away with V) transform their respective coordinates into each other, via the Lorentz transformation.

LT

Lorentz Transformation

$$\begin{aligned} ct' &= \gamma\left(ct - \frac{V}{c}x\right) \\ x' &= \gamma\left(x - \frac{V}{c}ct\right) \\ y' &= y \\ z' &= z \end{aligned} \tag{3.29}$$

with

$$\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{3.30}$$

We will look at the consequences for time and space coordinates.

Relativity of simultaneity

From the Lorentz transformation it is clear that time is not universal anymore ($ct' \neq ct$ in general). This is a large step from Newton and Galileo. Now the time coordinate is mixed somehow with the space coordinates depending on the speed V.

Let us consider 2 events in the reference frame of S;

- event A with coordinates (ct_1, x_1)
- and event B with (ct_2, x_2).

If the two events in S are simultaneous, i.e. $t_1 = t_2 \to ct_1 - ct_2 = 0$, then in S' they are in general not! Simultaneity is relative!

$$\begin{array}{rcl} {ct'}_1 & = & \gamma(ct_1 - \frac{V}{c}x_1) \\ {ct'}_2 & = & \gamma(ct_2 - \frac{V}{c}x_2) \\ \Rightarrow {ct'}_1 - {ct'}_2 & = & \gamma(ct_1 - ct_2) - \gamma\frac{V}{c}(x_1 - x_2) \end{array} \tag{3.31}$$

Even though the first term $(ct_1 - ct_2) = 0$ the second term $(x_1 - x_2)$ is never zero unless $x_1 = x_2$, and ${ct'}_1 - {ct'}_2 \neq 0$ in general.

In words: The events A and B that are simultaneous for S, are never simultaneous for S', unless the events are happening at the same place.

Relativität der Gleichzeitigkeit as Einstein called it, is the first very counterintuitive consequence by simple application of the Lorentz transformation. Our brains are not trained and build to cope with this aspect of nature. There is just no evolutionary advantage to it as all relevant speeds are much smaller than the speed of light.

Time dilation

We investigate how time intervals between a stationary and a moving observers are transformed. We can expect that these time intervals are not the same.

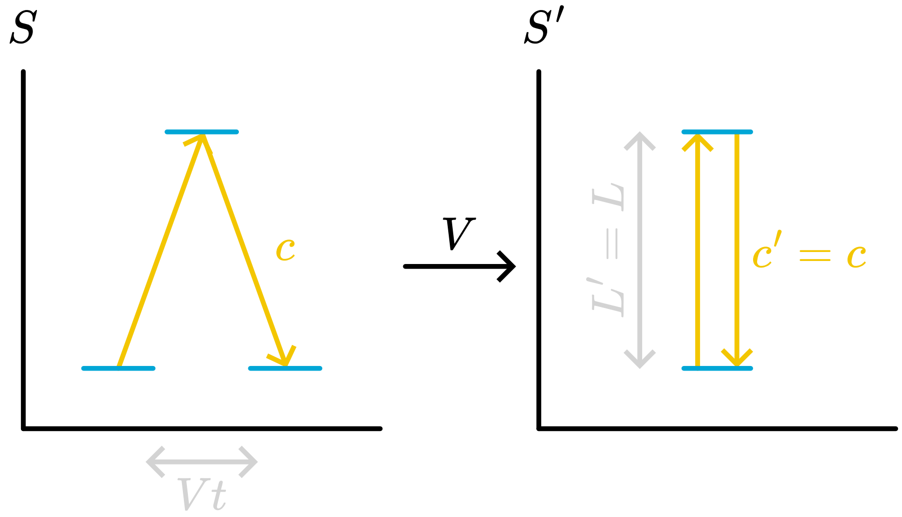


Figure 3.28: Clock stationary according to S' but moving for S.

If you consider the sketch above, we see how time intervals are counted for a moving observer and for an observer in the rest frame. A light ray is traveling between 2 mirrors. This up and down traveling of the light is a counter for the time. If you have never thought how time is measured, think a bit how a clock actually does that. Today, the second is defined as a (very large) number of tiny energy transitions (vibrations) of the Caesium-133 atom (see e.g. Atomic Clock).

Consider the time light travels for the observer S who sees the clock moving with velocity V. The clock counts one unit of time, t if the light has gone from the bottom mirror to the top one and back to the bottom mirror. Thus from bottom to top takes $t/2$. This means that the length of the light path from bottom mirror to top mirror is equal to $ct/2$ as light travels with velocity c. In that same period of time, the top mirror has moved a distance $Vt/2$, as the clock and thus the mirrors move with velocity V with respect to observer S. Now, we can relate the length of the light path from the bottom to the top mirror to the size of the clock, L and the displacement of the mirror, $Vt/2$: $L^2 + \frac{V^2}{4}t^2 = \frac{c^2}{4}t^2$ where we used Pythagoras, see Figure 8.

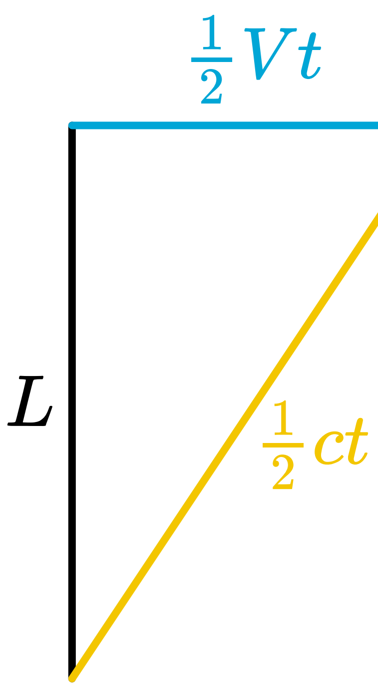


Figure 3.29: Light path in a moving clock.

We can solve this for the time t that the stationary observer S puts to the moving clock

$$t = \frac{2L/c}{\sqrt{1 - \frac{V^2}{c^2}}} = \gamma \frac{2L}{c} > \frac{2L}{c} \tag{3.32}$$

We see directly that the time the stationary observer S records is larger than the moving observer S' itself which is just $2L/c$ (the time in his rest frame)! The time interval gets longer/dilated by the γ-factor.

$$\Delta T = \gamma \Delta T_0 \tag{3.33}$$

with $\gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} > 1$ and T_0 the ***proper time*** or *eigen time* in the rest frame.

Note: a time interval is also the counting of your heart. That means the moving observer ages more slowly compared to the observer at rest. See the examples below for some experimental evidence of the time dilation.

Conclusion: moving clocks run slower, time gets stretched

Length contraction

The length of moving objects becomes smaller/contracted for the observer at rest. To explain this effect, we consider a moving rod with velocity V and with length L_0 in the rest frame.

Now that we have seen that time intervals are no longer universal, we need to think about:

$$what is it, measuring the length of an object? \tag{3.34}$$

Normally, we measure the length of an object by seeing how many times a measuring stick fits in the object. We obviously do this in the frame of reference in which the object doesn't move. There we don't need to worry about the moment we start at the left side of the object and arrive with our measuring stick on the right side. But if we would do so in a frame of reference in which the object is moving, that wouldn't work of course. By the time we would reach the right side of the object, it would no longer be at its starting position when we began our measurement and the number of times our ruler fits in the object is now influenced by the motion of the right side of the object.

To measure the length of a moving object, we thus need a different strategy. What we could do, is having a very long ruler fixed in our system. The object is moving past it. If we have two observers, one concentrating on the left side of the object and the other on the right side, we could ask them to measure the position of the left and right side of the object along the ruler **at the same time**. Then the difference of the left and right side on the ruler will give us the length of the object.

Thus: the length is measured from the difference of two events in space-time of the front and the back of the rod. We will call the events $E_L : (ct_1, x_1)$ and $E_R : (ct_2, x_2)$. As we measure size, we require: $t_1 = t_2$, that is the measurements are done simultaneously in S. According to S, the length of the rod is $L = x_2 - x_1$, nothing special here.

Next, we transform the events E_L and E_R to S':

$$\begin{aligned} x'_1 &= \gamma\left(ct_1 - \frac{V}{c}x_1\right) \\ x'_2 &= \gamma\left(ct_2 - \frac{V}{c}x_1\right) \end{aligned} \tag{3.35}$$

For S' the difference between x'_2 and x'_1 is of course the length of the rod. It doesn't matter for S' whether or not the coordinates the left and right side of the rod are measured at the same time. The rod is not moving in the frame of S'. Thus S' gets as length of the rod:

$$L_0 = x'_2 - x'_1 \tag{3.36}$$

with L_0 the **proper length** of the rod, i.e. the length according to an observer moving with the rod.

Now we invoke the Lorentz transformation for the two events E_L and E_R to find the relation between the coordinates used by the two observers:

$$L_0 = {x'}_2 - {x'}_1 = \gamma(x_2 - x_1) - \gamma\frac{V}{c}(ct_2 - ct_1) \tag{3.37}$$

As we measure x_1, x_2 at the same time in S, we have $ct_2 = ct_1$.

$$L_0 = \gamma(x_2 - x_1) = \gamma L \Leftrightarrow L = \frac{L_0}{\gamma} \tag{3.38}$$

The length of the moving object observed by the stationary observer is not the same as the length in the rest frame. The length observed by the stationary observer S gets smaller/ contracted by $\gamma > 1$ compared to the length in the rest frame of S': $L < L_0$.

Conclusion: moving rods are shorter, space shrinks

3.2.2.3 Paradox: twins and barns

There are many variants of the following paradox. The word *paradox* already implies that there is only an apparent contradiction, not a real one. Here we will formulate the paradox with a ladder & barn and resolve it, but you can also think about it as a train & tunnel, or tank & trench etc. The resolution is always the same.

Example: Barn & Ladder

Consider a ladder of rest length $L_l = 26\ \text{ m}$ and a barn of rest length $L_b = 10\ \text{ m}$. Obviously, the ladder does not fit in the barn, isn't it?

Now consider that the ladder is moving with velocity $V = \frac{12}{13}c\ \left(\gamma = \frac{13}{5}\right)$ towards the barn.

- For an observer in the barn, the length of the ladder is contracted to $L_l/\gamma = 26 \cdot \frac{5}{13} = 10\ \text{ m}$ exactly fitting in the barn which in her rest frame is $10\ \text{ m}$.
- For an observer moving with the ladder, the barn gets contracted to $L_b/\gamma = 10 \cdot \frac{5}{13} = 50/13 \sim 4\ \text{ m}$, being much too small to fit in the ladder. The ladder in his rest frame is $26\ \text{ m}$.

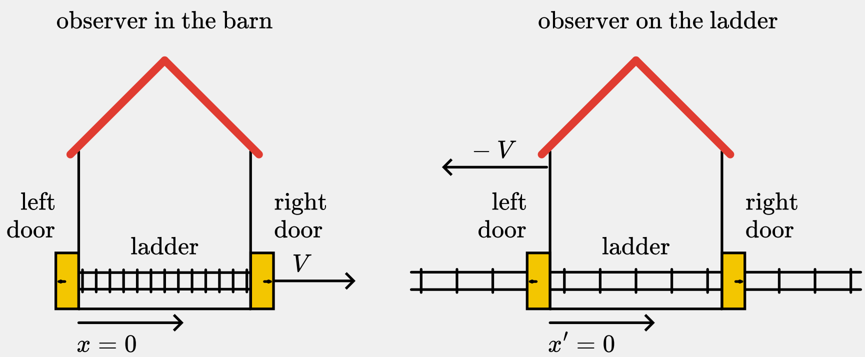


Figure 3.30: Ladder & Barn: perspective from two observers.

We have applied the Lorentz transformation or length contraction (time dilation) and the concept of relativity correctly, but something seems wrong! The physical outcome must be the same for both observers, but one observer claims the ladder perfectly fits into the barn, the other says it does not! That is: the observer in the barn can close the left and right door when the ladder is just inside the barn. Of course, the doors need to be open again very quickly as the ladder is moving with high velocity to the right. But that doesn't take away the fact that doors were closed and the ladder was inside the barn. How does the other observer cope with this?

You can have the same paradox not with length contraction, but time dilation, then it is called the *twin paradox*. We discuss the twin paradox later in the framework of Minkowski-diagrams.

Solution

The key to the resolution of the paradox is always the relativity of simultaneity. In this instance of the paradox with the barn and ladder: both observers are right but do not agree when the measurements are done.

Let's analyze the situation in detail using the Lorentz transformation. Later you can analyze it again qualitatively using a Minkowski-diagram which is quite insightful.

Our above "analysis" was a bit short: using length contraction. It is also a bit 'dangerous' as length contraction assumes simultaneous events in one frame.

We will consider how both observers would actually *measure* things in their respective frames of reference and in which order these happen. It turns out that both points of view are correct, but with a twist. We define 4 events to analyze the situation.

1. Event 1: right end ladder at left door barn
2. Event 2: right end ladder at right door barn
3. Event 3: left end ladder at left door barn
4. Event 4: left end ladder at right door barn (not really needed)

The four events are sketched in the figure below

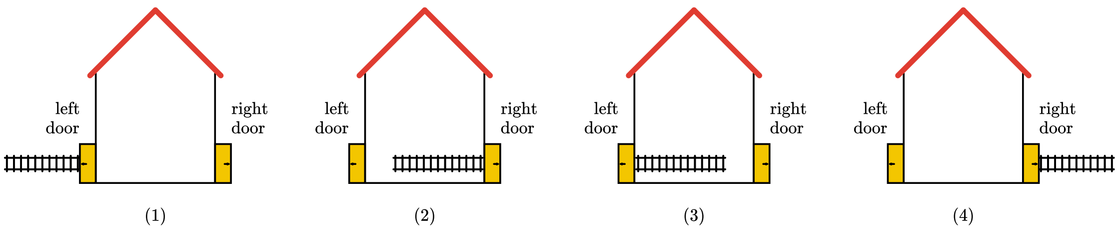


Figure 3.31: Four events of the ladder & barn paradox

Note: the size of the ladder in the sketch above is of course open for debate between the two observers :-).

Observer Barn (B) will conclude that the ladder fits inside the barn and actually is inside the barn if Event 3 is earlier than Event 2, according to the clock of observer B. If, however, Event 3 is later than Event 2, the ladder does not fit. Similarly, observer Ladder (L) will draw the same conclusions, but based on the clock of observer L.

Let's analyze these events. We will denote the coordinates of observer B as (ct, x) and those of observer L as (ct', x'). Both observers agree that they will call the position of the left door the origin, that is $x_{LD} = x'_{LD} = 0$. Moreover, they agree that at the moment the right end of the ladder is at the left door, they will set their clocks to 0. Remember: according to observe B, the length of the ladder is L_{0L}/γ = 10 m, which happens to be the size of the barn according to B. We anticipate that B will conclude that the ladder fits.

Next, we need to give the events their space-time coordinates, e.g. in the frame of Be and transform these coordinates according to the LT to the frame of L. This is done below, where we used: L_{0B} = proper length of barn, i.e. in the rest frame of the barn and L_{0L} = proper length of ladder, that is in the rest frame of the ladder. Note: $V/c = 12/13 \Rightarrow \gamma = 13/5$

Event	Barn (ct, x)	Ladder (ct', x')
1	$(0, 0)$	$(0, 0)$

Event	Barn (ct, x)	Ladder (ct', x')
2	$\left(\frac{c}{V} L_{0B}, L_{0B}\right)$	$\left(\frac{c}{V} \frac{L_{0B}}{\gamma}, 0\right)$
3	$\left(\frac{c}{V} L_{0B}, 0\right)$	$\left(\gamma \frac{c}{V} \frac{L_{0B}}{\gamma}, -L_{0L}\right)$

As we see, according to B, the left and right end of the ladder are exactly at the same moment at the left and right door of the barn, respectively (time coordinate of events 2 & 3 $ct_2 = ct_3 = \frac{c}{V} L_{0B}$). Consequently, observer B measures that the ladder (just) fits into the barn as anticipated by us. So B can close both doors and have the ladder inside the barn.

However, if we look at events 2 & 3 according to L, we see that L measures that the right end of the ladder is much earlier at the right door (event 2 $ct'_2 = \frac{c}{V} \frac{L_{0B}}{\gamma}$), than the left end is at the left door (event 3 $ct'_3 > ct'_2$). So, according to L, when the ladder hits the right end of the barn, the left part of the ladder is still left from the left door, thus outside the barn. The ladder does not fit. Of course, L sees that B closes the doors of the barn, but contrary to what B says: 'I closed the doors simultaneously and the ladder was in my barn“, L will respond: “that may be true for you, but I clearly observed that you first shut the right door, while the left was still open. Then you quickly opened the right door to let the ladder pass and only after a while, when the left side of the ladder was just inside your bar, you closed the left door. The ladder was never inside the barn with both doors closed!”

The paradox is, that both observers are right. Again we see demonstrated that simultaneous for one does not necessarily mean simultaneous for another. Very counter intuitive and yet: very true.

As you see both observers do not agree where the ladder is when the left door is closed. Where for the barn observer both doors close at the same time, this does not happen for the ladder observer.

Example: John Bell

This problem became known through John Bell.

Why you absolutely need to know John Bell

John Bell became famous by the inequalities that have his name attached. Bell's theorem from 1964 started to end (post mortem) the twist between Einstein and Bohr about quantum mechanics in favor for Bohr. In 1935 Einstein, Polodsky and Rosen came up with a paradox, named EPR paradox after their names, that seemed to show that quantum mechanics cannot be “complete” (.i.e *the real thing* describing reality). Bell's inequalities allowed to experimentally test who was right, and Einstein was fundamentally wrong. In 2022 the Nobel Prize in Physics was awarded to Clauser, Aspect and Zeilinger for their efforts to experimentally show that the Bell's inequalities are violated (and Bohr was right). In Delft Roland Hanson performed a *loophole-free Bell test* in 2015 which was big news.

Why is this so important? It touches the heart of what is reality, is it deterministic and/or local now that quantum mechanics turned out *to be* the real thing? How we see reality now boils down to how we interpret quantum mechanics - and that is difficult to comprehend. The Copenhagen interpretation is so frustrating as the wave function collapses at measurement, however, the many-world interpretation that avoids the collapse is also not very appealing as it needs an infinite number of universes. This remains one of the important open ends in physics.

In this thought experiment we have two spaceships B and C initially at rest and spaceship A as observer. B and C are connected by a tight but fragile string between them. A simultaneously signals B and C to accelerate equally, and B and C will have the same velocity at every time from the start.

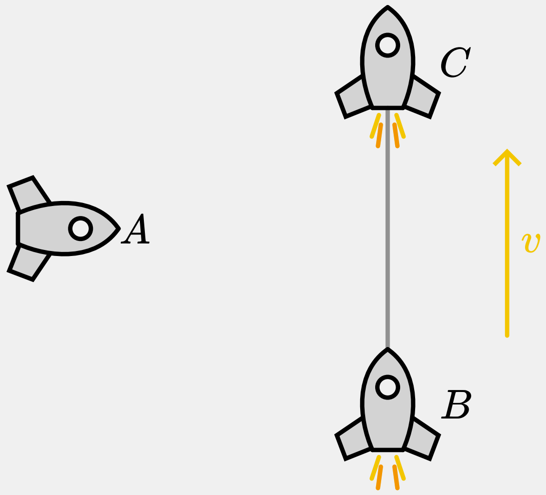


Figure 3.32: Bell's paradox: accelerating spaceships and a thin wire.

Question:

Will the string between B and C break eventually?

Answer

Yes.

Explanation:

One might think that the whole assembly of the two ships B and C and string undergo length contraction together, thus the string would not break, but that is incorrect.

- As seen from A's rest frame, B and C will have at every moment the same velocity, and so remain displaced one from the other by a fixed distance. The tying will not be long enough anymore due to length contraction and therefore break.
- The distance between B and C in the rest frame of B or C *increases* however as the acceleration from neither of them is simultaneous (if you work this out the relativity of simultaneity is the issue)! The thread breaks also in their frame.

If you got this wrong, do not worry, most people do (that is trained physicists).

If you think about this example for a bit, it becomes clear that relativistic acceleration is very troublesome for the structural integrity of extended objects! Another problem for our hopes of space travel to far away places.

3.2.3 Exercises, examples & solutions

Updated: 04 feb 2026

3.2.3.1 Examples

Example: Muon production in the upper atmosphere

Muons are elementary particles of the lepton family, the heavier brother of the electron. Muons decay via $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$ (or $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$. NB: You need the neutrinos to conserve lepton number) with a mean lifetime of $\tau = 2.2\ \ \mu$s. Muons are generated in the upper atmosphere (20 km) when a high energetic cosmic ray hits a nuclei as decay products. The speed of the muons is about $v = 0.99c$. If you compute velocity times lifetime $\tau v < 1$ km, then we conclude that nearly no muons should be detectable on the ground (assuming no other process interferes in the muons path). But we do? How is this possible?

Solution

You can solve this by considering the time dilation for an earth observer, as the lifetime is with respect to the rest frame! The lifetime for an earth observer is therefor stretched to $\gamma\tau \sim 16\ \ \mu$s. Therefore muons only need to travel about 4 lifetimes, and a decent fraction (1/16) can still be measured on the earth surface. You can also reason via length contraction of the path the muons travel 20 km$/\gamma$.

Example: Special relativistic correction to GPS timing

GPS uses satellites orbiting the earth at a lower altitude to determine the position. If you receive the signals from 4 or more satellites, you can compute your position by triangulation, e.g. measurement of time difference of the received signals. To this end you need a very precise timing of the signals. The satellites velocity is "slow" with $v = 4 \cdot 10^3$ m/s, and thus $\gamma \sim 10^{-5} \ll 1$. But the error in time measurement accumulates and due to time dilation even this small γ-factor will increase within 1 hour to a time error of 10^{-7} s or a position error of about 100 m. This would not be useful for navigation in a city and would required a recalibration of the system every few minutes. Later we see that a **general relativistic effect** is even more prominent!

Example: Relativistic correction to wavelength of electrons in a TEM

In a standard Transmission Electron Microscope the electrons are accelerated via electric potential differences of up to 300 kV. Assuming that particles have a wavelength via the idea of de Broglie $E = mc^2 = pc = h\frac{c}{\lambda} \Rightarrow \lambda = \frac{h}{p}$ we can use electrons as waves to image and magnify as with a normal light microscope. The smallest detail you can image with waves imaging in the far-field is given by the diffraction or Abbe resolution limit to $d \sim \frac{\lambda}{2}$. For microscopy with visible light ($\lambda \sim 500$ nm) this limit is a hard restriction. For electrons of low speeds we can use $\lambda = \frac{h}{mv}$, but for 300 kV acceleration the speed would be already larger than c! Later in the course you learn how to compute the **relativistic momentum**, filling in the numbers and the rest mass of the electron of 511 keV we obtain $\lambda \sim 2$ pm. About 10% *smaller* than from classical considerations. The diffraction limit to resolution is not an issue practically for the electrons as the distances between atoms in a solid are typically > 10 pm.

Exercise 3.33: Space expedition 🌶️

During their quest to find planets at other stars than our sun, ESA researcher discover a planet that shows striking similarities with earth. This planet orbits a star 40 lightyears from us. They start planning an expedition with astronauts. ESA requires that the astronauts upon arrival at the planet have aged no more than 30 years.

In this exercise, we ignore possible effects of acceleration. A lightyear is the distance traveled by a photon in one year.

1. What is the required velocity of the spaceship (with respect to the reference frame of the earth) to ensure a journey of 30 years (ignore the time spent on the other planet)?
2. What is according to the astronauts the distance the have to travel? Does that agree with the journey time of 30 years?
3. To inform Mission Control on earth the astronauts send yearly (according to their clock) a report to earth. Of course, the report is coded in the form of a light pulse. What is the period between receiving two consecutive reports according to Mission Control?

Exercise 3.34: 🌶️

An observer S' is traveling in a fast train. According to S', the train has a length $2L'$. The train is speeding with V over a track that is along the x-axis. At $t' = 0$ S' passes the origin of the frame of reference of S, who is stationary with respect to the track. At the moment of passing, S sets her clock to $t = 0$.

S' is in the middle of the train. He send at $t' = 0$ two light pulses out. One in the direction of the front of the train, where this pulse reflects on a mirror and is traveling back to S'. The other pulse is send to the back of the train and reflects there back to S'. S' claims that both pulses are received back at the same time.

1. Define the events that define this problem and give the coordinates as S' would do.
2. Translate the events to the frame of S.
3. Does S also see the two pulses reach $'$ at the same time?
4. Draw a (ct, x) diagram in which the the trajectories of S', front and back mirror as well as the two pulses are shown. Note: the ct-axis is the vertical axis in such a graph. Can you graphically understand whether or not the two pulses arrive at S' at the same time according to S.

Exercise 3.35: 🌶️

Observer S' is traveling with velocity V with respect to observer S. Both observers have aligned their x, x' axis and set their clocks to zero when their origins coincide.

According to S', an object is approaching S' at a velocity $-V$. At $ct' = 0$, the object is a distance L' from S'. At some moment in time it will collide with S'.

1. The initial time and position of the object at $ct' = 0$ is marked as Event 1 by S'. Provide the coordinates of E1 according to S' and according to S.
2. Determine the event "object collides with S'" (event E2) according to S' and according to S.
3. Can you understand the values of x_1 and x_2?

Exercise 3.36: 🌶️ 🌶️

Observer S' is traveling with velocity V/c=4/5 with respect to observer S. Both observers have aligned their x, x' axis and set their clocks to zero when their origins coincide.

According to S, an object is moving at a velocity -V/c = $-4/5$. At $ct = 0$, the object is in the origin of S. At some moment in time, ct, it is located somewhere on the negative x-axis.

Do the exercise twice: first for observers in the world of Einstein and Lorentz, second time for the world of Newton and Galilei.

1. Define two events: one (E1) where the object is at $ct = 0$ and the other (E2) where it is at ct. Transform both objects to S'.
2. For an object moving at constant velocity, the velocity can be found from two locations at two separate moments in time. Find the velocity of the object according to S' and show that its magnitude is smaller than the speed of light in the world of Lorentz and Einstein. To people living in the world of Newton and Galilei, this is a surprising result. They would have found a velocity magnitude larger than c.

Solution 3.37: Solution to Exercise 1

1. Denote Mission control by S and the spaceship by S'. According to S, the distance to the planet is $L = 40ly$. Thus the traveling time will be $\delta t_e = \frac{L}{V}$, with V the velocity of the spaceship according to S. S' time dilation $\rightarrow \delta t_e = \gamma \delta t_0$

Requirement: $\delta t_0 = 30ly \rightarrow \frac{L}{V} = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\delta t_0 \Rightarrow \frac{V}{c} = \frac{4}{5}$

1. Length contraction: $L' = \frac{L}{\gamma} \rightarrow L' = \frac{40}{5/3} = 24ly$

According to the astronauts, the planet is approaching them with a velocity $-V \Rightarrow \frac{V}{c} = -\frac{4}{5}$.
So they have to wait $\delta t'_w = \frac{L'}{\frac{4}{5}c} = 30y$

1. in S' a light pulse every year. Define event = nth pulse $(ct'_n, x') = (n, 0)$. The (n+1) pulse $(ct'_{n+1}, x'_{n+1}) = (n+1, 0)$ Transform to S via inverse LT

$$
\begin{aligned}
n^{th}pulse : \{ct_n &= \gamma\left(ct_n{}' + \frac{V}{c}x_n{}'\right) = \gamma ct_n{}' \\
x_n &= \gamma\left(x_n{}' + \frac{V}{c}ct_n{}'\right) = \gamma V t_n{}' \\
(n+1)^{th}pulse : \{ct_{n+1} &= \gamma\left(ct'_{n+1} + \frac{V}{c}x'_{n+1}\right) = \gamma ct'_{n+1} \\
x_{n+1} &= \gamma\left(x'_{n+1} + \frac{V}{c}ct'_{n+1}\right) = \gamma V t'_{n+1}
\end{aligned}
\tag{3.39}
$$

The nth arrives at earth after traveling the distance x_n with the speed of light. Hence, the moment of receiving is:

$$t_{n,e} = t_n + \frac{x_n}{c} = \gamma n\left(+\frac{V}{c}\right) \tag{3.40}$$

Similarly for the (n+1)th:

$$t_{n+1,e} = t_n + 1 + \frac{x_n + 1}{c} = \gamma(n+1)\left(+\frac{V}{c}\right) \tag{3.41}$$

So, we conclude that the time between receiving two consecutive pulses by Mission Control is:

$$\delta t_e = t_{n+1,e} - t_{n,e} = \gamma\left(+\frac{V}{c}\right) = 3year \tag{3.42}$$

Is that possible?

The astronauts send 30 reports while on their way to the planet as their journey to the planet takes 30 years. According to S this journey takes $\frac{L}{V} = 50year$. The last pulse is send 50 years after S' has left earth. This pulse need to travel 40ly and that takes 40 years. Thus it is received after 90 years. In those 90 years, 30 pulses have been received, hence Mission Control receives a pulse every 90/30 = 3 years.

This is a great example, that you need to be careful with quick answers based on time dilation. That would give $\gamma \cdot 1year = \frac{5}{3}year$ in between two pulses. But than we have forgotten that these pulses are not send from the same location.

Solution 3.38: Solution to Exercise 2

1. Events:

E0 - pulses send: $(ct'_0, x'_0) = (0, 0)$
E1R - forward traveling pulse hits front mirror: $(ct'_{1R}, x'_{1R}) = (L', L')$
E1L - backward traveling pulse hits back mirror: $(ct'_{1L}, x'_{1L}) = (L', -L')$
E2 - pulses send: $(ct'_2, x'_2) = (2L', 0)$

2. LT the events to S

E0: $(ct_0, x_0) = (0, 0)$
E1R: $(ct_{1R}, x_{1R}) = (\gamma(L' + \frac{V}{c}L', \gamma(L' + \frac{V}{c}L') = \gamma(1 + \frac{V}{c})L'$
E1L: $(ct_{1L}, x_{1L}) = (\gamma(L' + \frac{V}{c} - L', \gamma(-L' + \frac{V}{c}L') = \gamma(1 - \frac{V}{c})L'$
E2: $(ct_2, x_2) = \left(\gamma 2L', \gamma 2\frac{V}{c}L'\right)$

3. right pulse: first part of the traveling time is longer as the right mirror moves away, but on the way back S' approaches the pulse. The left pulse does exactly the opposite: first going to a mirror that is approaching and then moving after S' that is 'running away'.
4. This becomes evident in the (ct, x) diagram.

Figure 3.39: (x,ct) diagrams for S' and S)

Solution 3.40: Solution to Exercise 3

1. E1:

$$(ct'_1, x'_1) = (0, L') \Rightarrow \begin{cases} ct_1 = \gamma\left(ct'_1 + \dfrac{V}{c}x'_1\right) = \gamma\dfrac{V}{c}L' \\ x_1 = \gamma\left(x'_1 + \dfrac{V}{c}ct'_1\right) = \gamma L' \end{cases} \Leftrightarrow (ct_1, x_1) = \left(\gamma\frac{V}{c}L', \gamma L'\right) \tag{3.43}$$

2. trajectory object according to $S' \rightarrow$ linear motion with velocity $-V$: $x'(ct') = L' - \frac{V}{c}ct'$

collision with $S' \Rightarrow x'(ct'_2) = 0 \rightarrow ct'_2 = \frac{L'}{V/c}$

Thus, E2: $(ct'_2, x'_2) = \left(\frac{L'}{V/c}, 0\right)$

according to observer S:

$$\begin{aligned} ct_2 &= \gamma\left(ct'_2 + \frac{V}{c}x'_2\right) = \gamma\frac{L'}{V/c} \\ x_2 &= \gamma\left(cx'_2 + \frac{V}{c}ct'_2\right) = \gamma L' \end{aligned} \tag{3.44}$$

3. So, according to S the object hasn't moved! In retrospect, this is logical: S' sees S moving at velocity $-V$ and also sees the object moving at $-V$. Thus in S the object has zero velocity.

Note: we will come back to the transformation of velocities. That is more subtle than it know may look.

Solution 3.41: Solution to Exercise 4

Special Relativity with LT

1. E1: $(ct_1, x_1) = (0,0)$ en $(ct_2, x_2) = \left(ct, -\frac{V}{c}ct\right)$

LT naar S' with $\frac{V}{c} = \frac{4}{5}$ and $\gamma = \frac{5}{3}$:

$(ct'_1, x'_1) = (0,0)$

$$(ct'_2, x'_2) = \left(\gamma\left(ct - \frac{V}{c}\frac{-V}{c}ct\right), \gamma\left(-\frac{V}{c}ct - \frac{V}{c}ct\right)\right) = \left(\gamma\left(1 + \frac{V^2}{c^2}\right)ct, -2\gamma\frac{V}{c}ct\right) \tag{3.45}$$

2. velocity According to S: $\frac{v}{c} = \frac{x_2 - x_1}{ct_2 - ct_1} = \frac{-\frac{V}{c}ct}{ct} = -\frac{V}{c}$

According to S':

$$\frac{v'}{c} = \frac{x'_2 - x'_1}{ct'_2 - ct'_1} = \frac{-2\gamma\frac{V}{c}cdt}{\gamma\left(1 + \frac{V^2}{c^2}\right)ct} = -\frac{4/5}{1 + 16/25} = -\frac{40}{41} \tag{3.46}$$

Thus the magnitude of the velocity according to S' is less than c.

Newtonian mechanics with GT

1. E1: $(ct_1, x_1) = (0,0)$ en $(ct_2, x_2) = \left(ct, -\frac{V}{c}ct\right)$

GT:

$$\begin{aligned} ct' &= ct \\ x' &= x - \frac{V}{c}ct \end{aligned} \tag{3.47}$$

GT naar S' with $\frac{V}{c} = \frac{4}{5}$:

$$\begin{aligned} (ct'_1, x'_1) &= (0,0) \\ (ct'_2, x'_2) &= \left(ct, -\frac{V}{c}ct - \frac{V}{c}ct\right) = \left(ct, -2\frac{V}{c}ct\right) \end{aligned} \tag{3.48}$$

2. velocity According to S: $\frac{v}{c} = \frac{x_2 - x_1}{ct_2 - ct_1} = \frac{-\frac{V}{c}ct}{ct} = -\frac{V}{c}$ as before.

According to S':

$$\frac{v'}{c} = \frac{x'_2 - x'_1}{ct'_2 - ct'_1} = \frac{-2\frac{V}{c}ct}{ct} = -2\frac{V}{c} = -\frac{8}{5} \tag{3.49}$$

Thus the magnitude of the velocity according to S' is higher than c.

We will come back to this peculiar result in the world of Einstein and Lorentz.

3.2.3.2 Exercises

3.2.3.3 Answers

3.3 Velocity Transformation & Doppler shift

Updated: 04 feb 2026 Imagine we have two spaceships moving each with a speed of $\frac{3}{4}c$ as shown below. What is the speed that either the red or yellow spaceships sees for the other spaceship speed?

We should, first of all realize, that the information regarding the velocity of the two spaceships is given by an observer S who is neither in the red nor the yellow ship. We need to transform this information to an observer in the red or in the yellow ship.

Figure 3.42: Two spaceships approaching each other.

For the GT we have derived the velocity transformation to be

$$v'_{x'} = v_x - V \tag{3.50}$$

So, let's translate our velocity information from the observer S to someone in the red ship. The relative velocity between S and the red ship is $V = \frac{3}{4}c$. Thus according to the observer in the red ship, S_R, her velocity is $V'_R = V_R - V_R = 0$, obviously.

However, she will denote the velocity of the yellow ship as $V'_y = V_y - V_R = (-3/4 - 3/4)c = \frac{3}{2}c > c$. In the world of Galilei and Newton, this is no problem at all: velocities can be as big as you can imagine. However, in reality, this is not true. We have to use Special relativity if the velocities start to approach c. It is not possible for any object to move faster than the speed of light, as we will see later.

In the above, we have only looked at the velocity component in the x-direction. We have in addition found $v'_{y'} = v_y, v'_{z'} = v_z$.

As our universe does not follow Galilei and Newton, we need to derive the transformation/addition formula for velocities with the LT. So let's do it.

3.3.1 Velocity Transformation

Let us start from the definition of velocity (assuming we deal with constant velocities, so we don't need to worry about differentiation and integration). We will denote velocities by u to avoid confusion with V, the relative velocity between the two observers.

Observer S' will write down:

$$u'_{x'} = \frac{x'_2 - x'_1}{t'_2 - t'_1} = \frac{\Delta x'}{\Delta t'} \quad \textit{and} \quad u'_{y'} = \frac{y'_2 - y'_1}{t'_2 - t'_1} = \frac{\Delta y'}{\Delta t'} \quad (*) \tag{3.51}$$

We have left out the z'-component as that will be completely analogous to the y'-coordinate.

Observer S will use similar definitions. How do these observers translate velocity information that they get from each other?

We need to use the LT to transform (ct', x', y') to (ct, x, y):

$$\begin{aligned} x'_2 - x'_1 &= \gamma(x_2 - \tfrac{V}{c}ct_2) - \gamma(x_1 - \tfrac{V}{c}ct_1) \\ &= \gamma(x_2 - x_1) - \gamma\tfrac{V}{c}(ct_2 - ct_1) \\ \\ y'_2 - y'_1 &= y_2 - y_1 \end{aligned} \tag{3.52}$$

and

$$\begin{aligned} ct'_2 - ct'_1 &= \gamma\left(ct_2 - \tfrac{V}{c}x_2\right) - \gamma\left(ct_1 - \tfrac{V}{c}x_1\right) \\ &= \gamma(ct_2 - ct_1) - \gamma\tfrac{V}{c}(x_2 - x_1) \end{aligned} \tag{3.53}$$

From the last line it is clear that also the y, z components of the velocity $\vec{u}$ will be influenced by the transformation although the relative motion between the two observers is only along the x-direction. Substituting the expressions for the space and time difference into $v'_{x'}$, we obtain

$$\begin{aligned} u'_{x'} &= \frac{\gamma\Delta x - \gamma\frac{V}{c}\Delta ct}{\gamma\Delta ct - \gamma\frac{V}{c}\Delta x} = \frac{\frac{\Delta x}{\Delta t} - V}{1 - \frac{V}{c^2}\frac{\Delta x}{\Delta t}} \\ &= \frac{u_x - V}{1 - \frac{V u_x}{c^2}} \end{aligned} \tag{3.54}$$

For the transverse components y, z, we obtain due to the change of the time interval

$$u'_{y'} = \frac{\Delta y}{\gamma\Delta ct - \gamma\frac{V}{c}\Delta x} = \frac{u_y}{\gamma\left(1 - \frac{V u_x}{c^2}\right)} \tag{3.55}$$

In the limit of $u_x, V \ll c$ both formulas reduce to the Galilean Transformation as required. For $u_x \to c$ and $V \to -c$ the combined velocity will stay smaller than c. Check yourself that this is true.

The formula for the velocity transformation/addition are not so easy to remember. Later you will see how to derive them from the transformation properties of the 4-velocity, which is easy to remember.

For our example of the two approaching spaceships, $u_x = -\frac{3}{4}c, V = \frac{3}{4}c$ we find for the speed of the yellow approaching the red ship

$$u'_{x''} = \frac{-\frac{3}{4} - \frac{3}{4}}{1 + \frac{3}{4}\frac{3}{4}}c = -\frac{24}{25}c < c \tag{3.56}$$

This is again smaller (in absolute sense) than c. For the other ship we find of course the same, but with different sign.

3.3.2 Doppler effect

The Doppler effect is well-known from waves. You know it from daily life. If a car is passing you at high speed, the frequency of the sound you hear changes from approaching to driving away from you. The received frequency f_{obs} by you is higher than the emitted frequency f_{src} while the car is approaching, and smaller when it drives off.

Here we investigate the effect of a moving source that is emitting light with f_{src} (electromagnetic waves). This is one of the few cases where the relativistic case is easier than the classical effect. In the latter it matters if the source is moving or the medium. For EM-waves, however, there is no medium (aether) as we have seen which simplifies things.

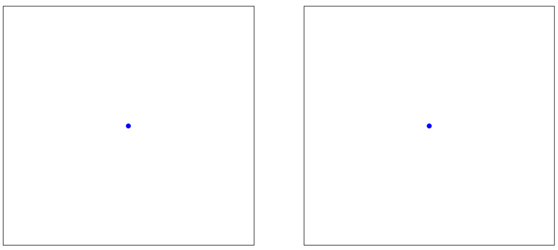


Figure 3.43: Effect on sound waves due to motion.

For the case of an observer with speed v_{obs} and speed of sound in the medium u and moving source v_{src} (e.g. car) the classical formula of the frequency shift is

$$f_{obs} = f_{src} \frac{u \pm v_{obs}}{u \mp v_{src}} \tag{3.57}$$

where for the stationary observer and medium, we have $+/-$ and for the moving observer and stationary source $-/+$.

The origin of the observed frequency shift of a moving source is visible in the picture. In the direction of motion, more wave maxima arrive per unit time, as the car is moving closer between two wave maxima. Consequently, the observer in the car will count more maxima per unit time: the frequency is higher.

For the relativistic effect we consider a moving source with velocity $\vec{u}$ moving with observer S' relative to S. The frequency of the source is $f_0 = \frac{1}{T_0}$ in the rest frame, with T_0 the period of the oscillation.

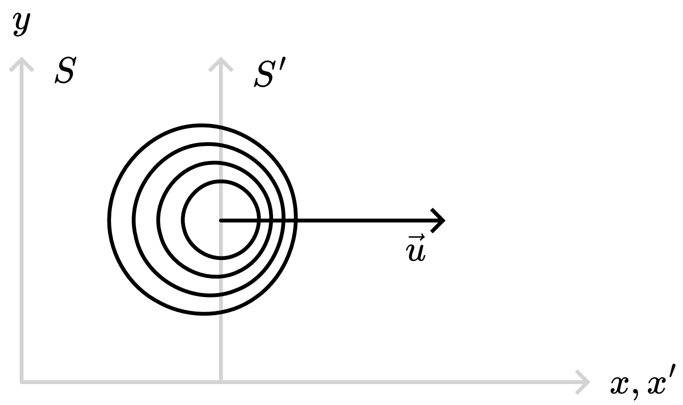


Figure 3.44: Observer S' and source both moving with respect to S.

We now consider the situation for S as shown in the figure below. The position of the source $\vec{r}_0$ is indicated with the star $*$.

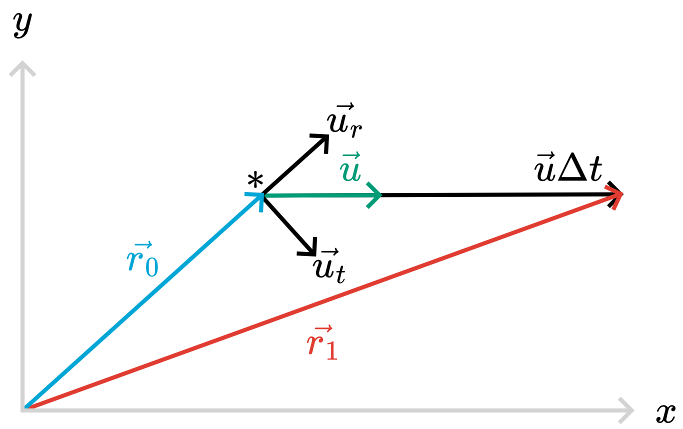


Figure 3.45: .

We do know that according to S', the proper frequency is f_0 and the proper period $T_0 = 1/f_0$. Thus if a maximum is send at t'_0 the next one will be at $t'_0 + \frac{1}{f_0}$.

S will denote the first maximum with time $t_1 = t_0$, but will have to take time dilation into account for the second one: $t_2 = t_0 + \frac{\gamma}{f_0}$. Note that these two time instants are the moments, according to S when the two maxima are send, not when they are received by S.

During this time interval $\frac{\gamma}{f_0}$ the source moves from $\vec{r}_0$ to $\vec{r}_1 = \vec{r}_0 + \vec{u}\frac{\gamma}{f_0}$. Thus, the distance that the second maximum has to travel is different from that of the first one, just like in the classical Doppler case.

We consider the 2 consecutive wave maxima that are emitted in S' and received in S:

- 1^{st} maximum in S' at ${t'}_0$, that is received in S at $t_1 = t_0 + \frac{r_0}{c}$. The additional time $\frac{r_0}{c}$ is needed for the light to travel from $\vec{r}_0$ to the observer at the origin of S.
- 2^{nd} maximum in S' at ${t'}_0 + \frac{1}{f_0}$, is received in S at $t_2 = \left(t_0 + \frac{\gamma}{f_0}\right) + \frac{r_1}{c}$.

To move further we split the velocity of the source into a radial component (in line to the observer in S) and a tangential part perpendicular $\vec{u} = \vec{u}_r + \vec{u}_t = u_r\hat{r} + u_t\hat{t}$. See Figure 5. If the distance $r_0 \gg u\frac{\gamma}{f_0}$ then the distance $r_1 = r_0 + u_r\frac{\gamma}{f_0}$.

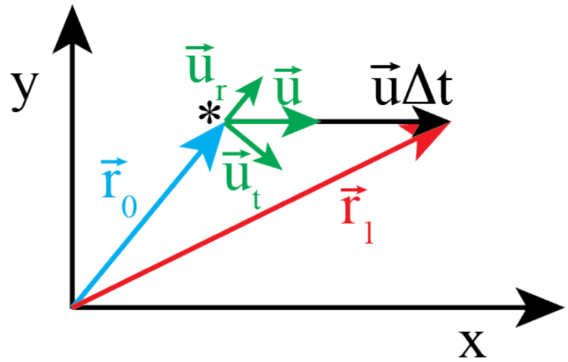


Figure 3.46: .

Note that we could drop the vector notation here from the exact relation above. Classically only the radial component is relevant as only the distance matters.

With this simplification on the distances we can compute t_2

$$t_2 = \left(t_0 + \frac{\gamma}{f_0}\right) + \frac{r_1}{c} \approx t_0 + \frac{\gamma}{f_0} + \frac{r_0 + u_r\frac{\gamma}{f_0}}{c} \tag{3.58}$$

For the frequency f in S we now subtract the two arrival times

$$\frac{1}{f} = t_2 - t_1 = \frac{\gamma}{f_0} + \frac{u_r}{c}\frac{\gamma}{f_0} \tag{3.59}$$

Rewriting this into a ratio of the emitted and received frequency, we obtain for the relativistic Doppler effect

$$\frac{f_0}{f} = \gamma\left(1 + \frac{u_r}{c}\right) = \frac{1 + \frac{u_r}{c}}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{3.60}$$

Please observe that the γ factor is of $\gamma(u)$ that means even for only tangential movement $(u_r = 0)$ there is a Doppler shit.

3.3.2.1 Cosmic background radiation

The most well-known frequency shift is the red-shift from the expanding universe. Light coming from a star has a certain frequency as observed in the rest frame of the star. This frequency corresponds to a quantum mechanical drop in energy level. The photon emitted in this energy transition has a well known wave length and frequency. That is, of course, in the rest frame of the process. Anybody observing these photons while traveling with respect to the star will see a Doppler-shifted frequency. If the star is moving away from the observer, the wave length seems longer and thus the frequency lower: the light is shifted to the red end of the spectrum. If on the other hand the star is moving towards the observer, the waves seem compressed and the frequency detected is higher: the light is shifted to the blue end of the spectrum. Hence, astronomers talk about red sifted and blue shifted light.

The astronomer Edwin Hubble first found in the 1920s that the universe does not only consists out of our own galaxy, the milky way, but there must be (many) other galaxies, which were called *nebula* at that time. Second he could show that all further away galaxies move away from us by measuring the Doppler shift of well-known emission lines of stars and their distance from periodic intensity variation of Cepheid Variable stars. It turned out that the distance of the galaxies d was roughly linearly proportional to the red-shift which is again linearly related to the the radial velocity v as we derived. This is known now as Hubble's law $v = H_0 d$ with the Hubble constant ($H_0 \sim 70 \quad km/s/Mpc$). Further away galaxies move faster away, but why? And why is no galaxy approaching us?

At end of the 1920s Georges Lemaître applied Einstein general theory of relativity to cosmology and found that the universe must be expanding, while it started in a "primeval atom", now known as the *Big Bang*. He could explain the red-shift relation from the expanding universe hypothesis.

The Big Bang theory was highly debated early on, in particular by Einstein, but is now fully accepted. The strongest experimental evidence was the discovery of the *cosmic background radiation* in 1965 (by accident).

The whole cosmos is nearly uniformly filled by a background radiation of about $2.7 \quad \mathrm{K}$ (wavelength in the μm range) with small inhomogeneities as shown in the picture by the Planck satellite around 2013.

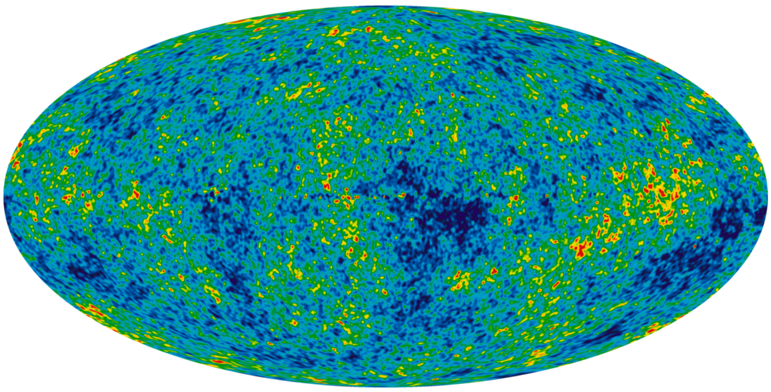


Figure 3.47: Background radiation in the universe as observed from earth. By NASA / WMAP Science Team - http://map.gsfc.nasa.gov/media/121238/ilc_9yr_moll4096.png, Public Domain.

This radiation is the red shifted radiation from around 380,000 years after the Big Bang when the universe became transparent. At that time the temperature had dropped (due to the adiabatic expansion) to around 3000 K, at which protons and electrons can form stable hydrogen atoms $p + e^- \rightarrow H$. This event is called *recombination*. At higher temperatures photons are scattered from the free electrons (and protons) constantly, effectively the photons have a very short mean free path and the universe is opaque. At the recombination temperature all of a sudden the photon could travel without strong scattering, the universe was transparent. The 3K cosmic background radiation that we measure today is the red-shifted version of this 3000 K light. It gives us a glimpse of how the universe looked at that time. Apart from the background radiation there were no other light sources in the universe as stars had not formed yet, the Dark Ages of the universe began.

The red-shift here is actually caused by the expansion of the universe itself (the universe expands causing the photons' wavelength to expand). NB: Time in cosmology is often given in units of red-shift (e.g. the red-shift for recombination is $z = 1089$).

Wavelength temperature relation

How can we relate the wavelength of electro-magnetic radiation to temperature?

Matter emits electro-magnetic radiation depending on its temperature. This relation is given by Planck's law with which quantum mechanics started in 1900 as he considered *black body radiation*. The emitted spectral density per solid angle depends on the thermal energy kT and is given by

$$u(\lambda, T) = \frac{2hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \tag{3.61}$$

Here for the first time h, Planck's constant, was introduced to quantize energy packages hf of oscillation.

Phenomenologically, the relation between the peak of the spectrum and the temperature was found by Wien already earlier to be $\lambda_{peak} = \frac{b}{T}$ with b Wien's constant $b \sim 2900 \quad \mu m \cdot K$.

NB: If you buy a light bulb for a lamp, then a temperature is indicated on the packaging, e.g. 2700 K for "warm white", 4000 K for "cool white" to describe the light color. Quantum physics at your local super market.

3.3.3 Examples, exercises and solutions

Updated: 04 feb 2026

3.3.3.1 Examples

Example: 21cm hydrogen line

21 cm line of hydrogen in radio astronomy.
The proton and electron in the hydrogen atom both have a magnetic dipole moment related to their spin. The total quantum mechanical wave function can have 2 states for the spins, parallel or anti parallel, where anti parallel is energetically favorable. The transition between these two states is forbidden to first order (you will learn more about this in the three courses on *Quantum mechanics* in the second and third year). By Fermi's golden rule of quantum mechanics that means the probability that it happens per second is very small, here $10^{-15}s^{-1}$ or that the lifetime of the excited state is very long $\sim 10^7$ years. As space is vast and there is much hydrogen, however, this still happens a lot such that we can observe it.

Due to the very long life time, the emission line is very sharp, i.e. it has a small natural spectral broadening. This can be seen from Heisenberg's uncertainty principle $\Delta E \Delta t \sim \hbar$. If Δt is very large, then ΔE is small and the spectral line is very sharp. Therefore shifts to this line must come from Doppler shifts which can be used to measure speeds accurately.

Exercise 3.48: 🌶️

Observers S' is moving at $V/c = 3/5$ with respect to S. Both observers have their x and x' axis aligned. If they are at the same position ($x = x' = 0$), they set their clocks to zero.

S' observes an object traveling at $4/5$ of the speed of light in the negative x'-direction.

Calculate the velocity according to S.

Exercise 3.49: 🌶️

Same situation as in Exercise 1, but now S' observes that the object is moving in the y'-direction with velocity $\frac{4}{5}c$.

Show that the magnitude of the velocity of the object according to S is smaller than c.

Exercise 3.50: 🌶️

In order to send information via electro-magnetic waves, people use amplitude modulation (AM) and frequency modulation (FM). AM means that the amplitude of the wave that is send out varies: the variations can be detected by the receiver and 'decoded' to the message. FM, on the other hand, means that the frequency of the wave is changing. This can also be detected and decoded to the message.

Captain Kirk on board of the starship USS Enterprise is traveling at a speed of $\frac{V}{c} = \frac{40}{41}$ with respect to earth. He uses FM and sends his monthly report to mission control using a center frequency of 10GHz. What is the frequency that Mission Control needs to look for in case:

1. Enterprise is moving straight towards earth;
2. Enterprise moves radially away from earth;
3. Enterprise moves tangentially to earth.

Exercise 3.51: 🌶️ 🌶️

In the year 2525 a young Applied Physics student (who doesn't take his study too seriously) is caught ignoring a red traffic light and gets a fine. Trying to be a smarty, he refuses to pay and calls for a hearing in court.

The judge asks the student why he doesn't want to pay: ignoring a red traffic light is dangerous and a fine is in place.

The student argues, that he wasn't ignoring a red light: the light was clearly green. The judge asks: "which light: the bottom one, the middle one or the top one?"

The students replies: the top one of course. I was riding my fat bike at a lovely high speed and noticed that only the top light of the traffic light was on. And it was definitely green."

The judge has heard enough. She adjourns the session and retreats to her office. There, she picks up her notebook and calculates what the velocity of the student was. Then she calculates the fine for speeding, using the formula "fine = 5Euro * (speed (in km/h) - 40km/h)".

She returns to the court room and the session is continued by her ruling. The student is acquitted of running a red light but is fined for speeding.

What is the amount of the fine?

Solution 3.52: Solution to Exercise 1

According to S' the object has velocity $u'_{x'}/c = -4/5$. Observer S uses the velocity transformation for the x-component of velocities:

$$\frac{u_x}{c} = \frac{\frac{u'_{x'}}{c} + \frac{V}{c}}{1 + \frac{V}{c}\frac{u'_{x'}}{c}} = \frac{\frac{-4}{5} + \frac{3}{5}}{1 - \frac{3}{5} \cdot \frac{4}{5}} = -\frac{5}{13} \tag{3.62}$$

Solution 3.53: Solution to Exercise 2

According to S' the object has velocity $u'_{x'} = 0$ and $u'_{y'}/c = 4/5$. Observer S uses the velocity transformation for the x and y-component of velocities:

$$\frac{u_x}{c} = \frac{\frac{u'_{x'}}{c} + \frac{V}{c}}{1 + \frac{V}{c}\frac{u'_{x'}}{c}} = \frac{3}{5} \tag{3.63}$$

$$\frac{u_y}{c} = \frac{\frac{u'_{y'}}{c}}{\gamma(V)\left(1 + \frac{V}{c}\frac{u'_{x'}}{c}\right)} = \frac{\frac{4}{5}}{\frac{5}{4}} = \frac{16}{25} \tag{3.64}$$

Thus the magnitude of $\vec{u}$ is:

$$\frac{u}{c} = \sqrt{\frac{u_x^2}{c^2} + \frac{u_y^2}{c^2}} = \sqrt{\frac{481}{625}} \approx \frac{22}{25} < 1 \tag{3.65}$$

Solution 3.54: Solution to Exercise 3

Doppler shift

$$\frac{f_0}{f} = \frac{1 + \frac{u_r}{c}}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{3.66}$$

In this case: $u/c = 40/41 \rightarrow \gamma = \frac{41}{9}$

1. $u_r/c = -40/41 \rightarrow \frac{f_0}{f} = \frac{1}{41}\frac{41}{9} \rightarrow f = 9f_0 = 90\text{GHz}$
2. $u_r/c = 40/41 \rightarrow \frac{f_0}{f} = \frac{81}{41}\frac{41}{9} \rightarrow f = \frac{1}{9}f_0 = 1.11\text{GHz}$
3. $u_r/c = 0 \rightarrow \frac{f_0}{f} = \frac{41}{9} \rightarrow f = \frac{9}{41}f_0 = 2.20\text{GHz}$

3.3.3.2 Exercises

3.3.3.3 Answers

Solution 3.55: Solution to Exercise 4

Obviously, the student tries to claim that due to his high speed, the red color of the traffic light was green to him. As he is approaching the light source, with a velocity V/c, he may also take the point of view of an observer in a frame in which he is not moving, but the traffic light is approaching with V/c,

The wave length of red light is 630nm and of green light 530nm. Or in terms of the corresponding frequencies: $f_r = \frac{c}{\lambda_r} = 4.76 \cdot 10^{14}\text{Hz}$ and $f_g = 5.66 \cdot 10^{14}\text{Hz}$. In the rest frame of the traffic light, the frequency is thus: $f_0 = f_r$, whereas in the frame of the student it is $f = f_g$.

If we plug this into the Doppler shift formula, we get:

$$\begin{gathered}\frac{f_0}{f} = \frac{f_r}{f_g} = 0.82 = \frac{1 + \frac{u_r}{c}}{\sqrt{1 - \frac{u^2}{c^2}}} = \sqrt{\frac{1 + V/c}{1 - V/c}} \Rightarrow \\ \frac{1 + V/c}{1 - V/c} = 0.68 \rightarrow \frac{V}{c} = 0.2\end{gathered} \tag{3.67}$$

Thus the biker claims to have a speed of 20% of the speed of light, that is 2.1610^8km/h and accordingly gets a fine of 1.08 billion Euro.

3.4 Spacetime and 4-vectors

Updated: 04 feb 2026

3.4.1 Space time

In 3D space we define a point/coordinate by its components (x, y, z) where all components have the same unit. We can do this also in 4D space time by an event (ct, x, y, z) as ct has unit length (it should be called *time space* by this ordering, but what ever). The same unit for all components is needed if we want to do geometry with the coordinates.

If we want to measure distances Δs between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) we do this in 3D Euclidean space as $\Delta s^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$. These distances are Galileo invariant, observer S and S' moving with $\vec{V}$ measure the same distance $\Delta s^2 = \Delta s'^2$. Note, that we take these two pints at the sam time t: $t_1 = t_2$. Or rephrased: we perform the measurement in the rest frame of the object we measure. That makes sense: measuring the length of an object that is moving requires that we measure the left and right side at the same time. Otherwise, the motion of the object will interfere with our measurements of the length.

The above statement is easily shown by invoking the Galilean Transformation:

$$\begin{aligned} x' &= x - Vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{3.68}$$

We transform the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) at the same time t, we get:

$$\begin{aligned} {x'}_1 &= x_1 - Vt, {x'}_2 = x_2 - Vt \Rightarrow {x'}_2 - {x'}_1 = x_2 - x_1 \\ {y'}_1 &= y_1, {y'}_2 = y_2 \Rightarrow {y'}_2 - {y'}_1 = y_2 - y_1 \\ {z'}_1 &= z_1, {z'}_2 = z_2 \Rightarrow {z'}_2 - {z'}_1 = z_2 - z_1 \\ t' &= t \end{aligned} \tag{3.69}$$

If we want to measure distances in space time and require that the distance is now Lorentz invariant, we cannot measure distance the same way! If we measure in S the positions at the same time, that will in general be at different times according to S'. Time is relative!

To do geometry, measure angles etc. we need an inner product and the inner product provides a distance measure (a metric) by the norm. For 3D you know that for two vectors $\vec{r}_1$ and $\vec{r}_2$: $\Delta s^2 = | \, | \, \vec{r}_1 - \vec{r}_2 \, | \, |^2 = (\vec{r}_1 - \vec{r}_2) \cdot (\vec{r}_1 - \vec{r}_2) = \Delta x^2 + \Delta y^2 + \Delta z^2$. Clearly the inner product in 4D space time cannot be defined in the same way.

We want that two relativistic observers measure the same distance (e.g. between two events), that is, it must be Lorentz invariant. We start by noting that the speed of light is constant for both observers. A light wave traveling in S and S' must therefore obey

$$c^2t^2 - x^2 - y^2 - z^2 = 0 = c^2t'^2 - x'^2 - y'^2 - z'^2 \tag{3.70}$$

Given this observation it is needed (and natural) to define the distance in space time as

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{3.71}$$

Warning

Notice directly that the distance Δs^2 can be negative! (And we are OK with that).

It is straightforward to show that the above distance ds^2 is indeed a Lorentz Invariant, i.e. $ds'^2 = ds^2$. Suppose we have two events: $E_1 : (ct, x, y, z)$ and $E_2 : (c(t + dt), x + dx, y + dy, z + dz)$. We can transform these to S' via the standard Lorentz Transformation:

$$
\begin{array}{rclcrcl}
ct'_2 & = & \gamma\big(c(t+dt) - \frac{V}{c}(x+dx)\big) & \Rightarrow & cdt' & = & \gamma\big(cdt - \frac{V}{c}dx\big) \\
x'_2 & = & \gamma\big((x+dx) - \frac{V}{c}c(t+dt)\big) & \Rightarrow & cdx' & = & \gamma\big(dx - \frac{V}{c}cdt\big) \\
y'_2 & = & y_2 & \Rightarrow & dy' & = & dy \\
z'_2 & = & z_2 & \Rightarrow & dz' & = & dz
\end{array}
\tag{3.72}
$$

Clearly, we do only have to concentrate on the cdt and dx terms:

$$
\begin{aligned}
cdt'^2 - dx'^2 &= \gamma^2\big(cdt - \tfrac{V}{c}dx\big)^2 - \gamma^2\big(dx - \tfrac{V}{c}cdt\big)^2 \\
&= \gamma^2\Big(c^2dt^2 - 2\tfrac{V}{c}cdtdx + \tfrac{V^2}{c^2}dx^2 - dx^2 + 2\tfrac{V}{c}dxcdt - \tfrac{V^2}{c^2}c^2dt^2\Big) \\
&= \underbrace{\gamma^2\Big(1 - \tfrac{V^2}{c^2}\Big)}_{=1}(c^2dt^2 - dx^2) \\
&= c^2dt^2 - dx^2
\end{aligned}
\tag{3.73}
$$

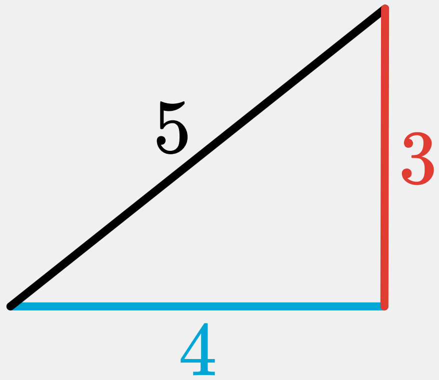
Note that if we had used a + sign, that is $ds^2 \equiv c^2dt^2 + dx^2$, we would **not** have arrived at a Lorentz Invariant.

Example: Pythagoras gets mixed up

We are used to all kind of 'obvious' results that hold in our Galilei/Newtonian world. For instance, for a triangle with a perpendicular angle we can apply Pythagoras theorem:

$$a^2 + b^2 = c^2 \tag{3.74}$$

Example: for a triangle with sides $(3, 4, 5)$ this would give the figure below.



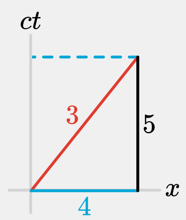
How does this work in our Lorentz/Einstein world?

Consider the following: according to S, a particle is moving with velocity $\frac{V}{c} = \frac{4}{5}$ over the x-axis. The particle is at $ct = 0$ at $x = 0$. Obviously, $5ls$ later it is at position $x = 4$. So, we can define two events:

$$
\begin{aligned}
E1 : (ct_1, x_1) &= (0, 0) \\
E2 : (ct_2, x_2) &= (5, 4)
\end{aligned}
\tag{3.75}
$$

Can we draw this? Sure, now we need an (ct, x) diagram. It is a convention to draw the ct-axis vertically.

The figure is going to look like this.

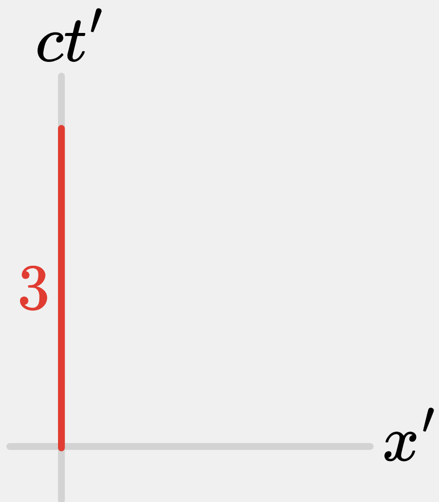


Much to our surprise, the hypotenusa is shorter than each of the other two sides!

Why does this make sense? In the world of Special Relativity, we can find answers by looking at a different frame of reference. What will observer S', who is traveling with the particle, say about this?

We have to translate the two events of E_1 and E_2 to the frame of S':

$$\begin{aligned} E1 : ({ct'}_1, {x'}_1) &= (0,0) \\ E2 : ({ct'}_2, {x'}_2) &= \left(\gamma\left(ct_2 - \frac{V}{c}x_2\right), \gamma\left(x_2 - \frac{V}{c}ct_2\right)\right) \\ &= \left(\frac{5}{3}\left(5 - \frac{4}{5}4\right), \frac{5}{3}\left(4 - \frac{4}{5}5\right)\right) \\ &= (3,0) \end{aligned} \tag{3.76}$$



Of course, as we knew, the length of the interval stays the same: $\Delta s^2 = \Delta s'^2 = 3^2$.

3.4.2 4-vector

The idea of having to work with a 'position' vector with 4 components with an inproduct as we discussed above, is generalized to vectors, i.e. quantities with a direction and a magnitude.

We define a 4-vector $\vec{A} = A^\mu = (A^0, A^1, A^2, A^3)$ to be a vector that transforms between two observers S and S' moving with V along the x-direction by the LT

$$\begin{aligned} A^{0\prime} &= \gamma\left(A^0 - \tfrac{V}{c}A^1\right) \\ A^{1\prime} &= \gamma\left(A^1 - \tfrac{V}{c}A^0\right) \\ A^{2\prime} &= A^2 \\ A^{3\prime} &= A^3 \end{aligned} \tag{3.77}$$

Other tuples of 4 values are not 4-vectors. The requirement that the 4-vector must transform via the LT is essential. We will use this later for the 4-velocity and 4-momentum.

3.4.2.1 Inner product & conventions

Like the distance also the inner product can be defined between two *4-vectors*. We use a capital letter for a 4-vector

$$\vec{A} = A^\mu = (A^0, A^k) = (A^0, A^1, A^2, A^3) = (A^0, \vec{a}) \tag{3.78}$$

This notation is just to a make clear distinction with 3-vectors that only have spatial coordinates. With a Greek index μ, A^μ we indicate all 4 components of the vector, while with a Latin index k, A^k we only indicate the spatial components. We also start counting at 0 for the first component, which is 'time'.

The inner product between two 4-vectors $\vec{A}, \vec{B}$ is now defined according to the rule we already saw before

$$\vec{A} \cdot \vec{B} \equiv A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3 \tag{3.79}$$

This is not a "choice" for the inner product, but follows strictly from the requirement that distance or length should not change under LT. A space with this inner product is called *Minkowski space* or the space has a *Minkowski metric* after Hermann Minkowski.

Notice that time component $(+)$ is treated differently than the spatial components $(-)$ in the inner product. Sometimes the inner product is also called *pseudo Euclidean* as there are −1 and +1 present in the inner product (instead of only +1 for Euclidean space).

3.4.2.2 Lorentz invariants

As is clear by the above construction the inner product of two 4-vectors must be LT invariant, that is for observers $S : \vec{A}, \vec{B}$ and $S' : \vec{A}', \vec{B}'$ it holds

$$\vec{A} \cdot \vec{B} = \vec{A}' \cdot \vec{B}' \tag{3.80}$$

This property can be a *very* powerful tool (OK, we constructed it that way). If we know the value of the inner product in one frame of reference, it will be the same in all other inertial frames of reference! We will use that later often. It is also clear that the distance interval ds^2 is a Lorentz invariant.

Inner product LT invariant: the hard way

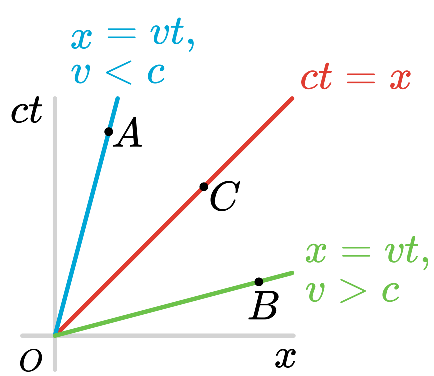
If you do not believe that the inner product is LT invariant you can write it out of course (with $\beta \equiv \frac{V}{c}$, a short hand notation that is frequently used).

We compute $\vec{A}' \cdot \vec{B}'$. We will concentrate on only $A^0 B^0 - A^1 B^1$, as with the standard Lorentz Transformation the A^2 and A^3 component are trivial.

$$\begin{aligned}
\vec{A}' \cdot \vec{B}' &= \gamma(A^0 - \beta A^1) \cdot \gamma(B^0 - \beta B^1) - \gamma(A^1 - \beta A^0) \cdot \gamma(B1 - \beta B^0) \\
&= \gamma^2 (A^0 B^0 - \beta A^1 B^0 - \beta A^0 B^1 + \beta^2 A^1 B^1) \\
&- \gamma^2 (A^1 B^1 - \beta A^0 B^1 - \beta A^1 B^0 + \beta^2 A^0 B^0) \\
&= \gamma^2 (1 - \beta^2)(A^0 B^0 - A^1 B^1) \\
&= A^0 B^0 - A^1 B^1 \\
&= \vec{A} \cdot \vec{B}
\end{aligned} \tag{3.81}$$

3.4.3 The light cone

Let us consider an event in space time $\vec{X} = X^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$. For sake of simplicity we only consider one space like component here. In the sketch we have the space axis (x or x^1) to the right and the time axis (ct or x^0) up. We consider 3 events A, B, C (points in space time) and their connection to the origin O

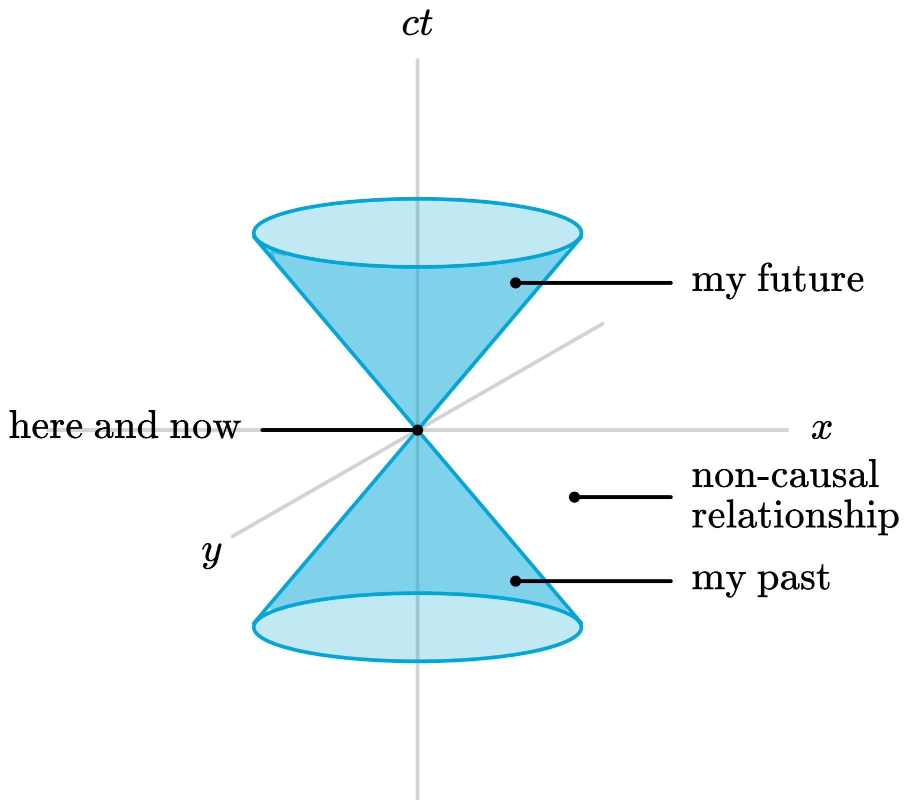


- OA: The point A can be reached from O with velocity $v < c$, therefore it is called *causally connected* or *time like*. For the distance $OA : \Delta s^2$, we see from projection of the coordinates A onto the time and space axis $\mid x_A - 0 \mid < (ct - 0) \Rightarrow \Delta s^2 > 0$.

Because the time component is larger than the space component, it is called *time like*. The distance is positive.

- OB: The point B can be reached from O only with velocity $v > c$, therefore it is called *non-causally connected* or *space like*. For the distance $OB : \Delta s^2$, we see from projection of the coordinates B onto the time and space axis $\mid x_B - 0 \mid > (ct - 0) \Rightarrow \Delta s^2 < 0$. Because the space component is larger than the time component, it is called *space like*. The distance squared is negative.
- OC: The point C can be reached from O only with velocity $v = c$, therefore it is called *light like* or *null*. For the distance $OB : \Delta s^2$, we see from projection of the coordinates C onto the time and space axis $\mid x_C - 0 \mid = (ct - 0) \Rightarrow \Delta s^2 = 0$. Because the space component is equal to the time component, it is called *light like*. The distance is zero. Therefore it is also called *null*.

Here you visually can observe that the sign of the distance using the Minkowski inner product classifies parts of space time.

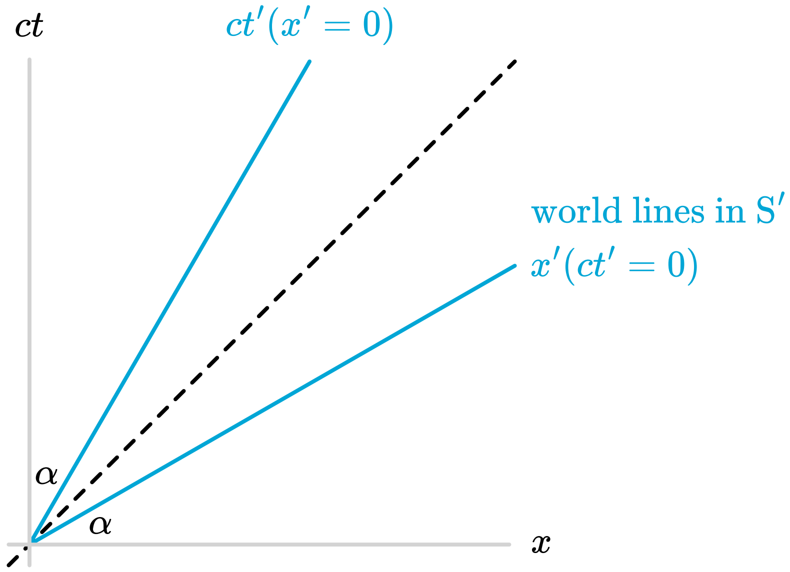


This is even more evident if you look at the light cone in the sketch. The cone mantel is generated by revolving the line $x = ct$, a light line. Here only a 2D cone is shown (ct, x, y), but of course this should be a 3D cone (ct, x, y, z). The inside of the cone at negative times is the *past* that could have influenced me at *now*. My *now* can influence my *future* (inside the cone to positive times). All the rest, outside the cone is not causally connected to me.

3.4.4 Minkowski-diagram

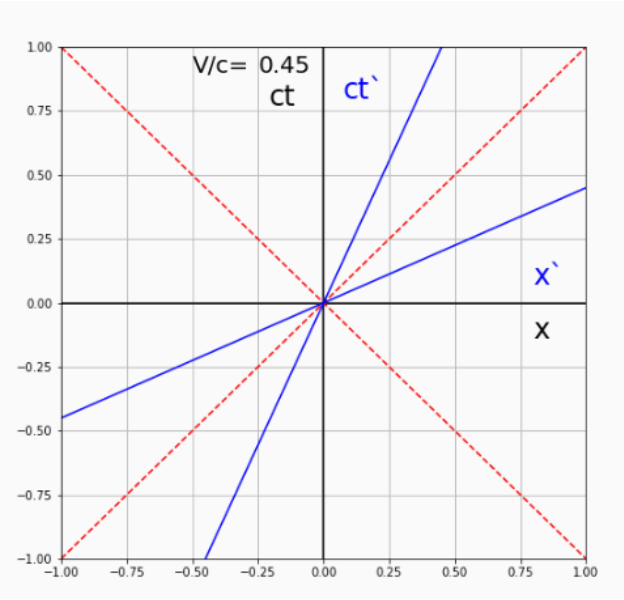
Now we can have a look at world lines of an observer S' with respect to S traveling with V along the x —axis in a graphical manner. The world line of an object is the path that an object travels in the 4-dimensional spacetime.

We plot the coordinate system of S' (blue) in the coordinate system of S (black).



- The time line of S' in S is given by the fact that $x' = 0$. From the LT we have $x' = \gamma\left(x - \frac{V}{c}ct\right) = 0 \Rightarrow x = \frac{V}{c}ct$. The angle α of the ct'-line with the ct axis is given by $\tan\alpha = \frac{V}{c}$.
- The space line of S' in S is given by the fact that $ct' = 0$. From the LT we have $ct' = \gamma\left(ct - \frac{V}{c}x\right) = 0 \Rightarrow ct = \frac{V}{c}xt$. The angle α of the x'-line with the x axis is given by $\tan\alpha = \frac{V}{c}$.

Both lines of S' make the same angle α with the coordinates axis of S. They lie symmetric around the light line $x = ct$ (diagonal with $\alpha = 45$ deg). The higher the speed V the higher the angle and the closer the lines lie to the light line. See the animation below, where the (ct', x') axis are plotted in the (ct, x) diagram of S for different values of V/c.



To further investigate how this plot can help us, let us consider lines of equal time in S. These are just the lines perpendicular to the ct-axis, and parallel to the x-axis, as you expect. And of course, lines parallel to ct, perpendicular to x are lines of constant space coordinate.

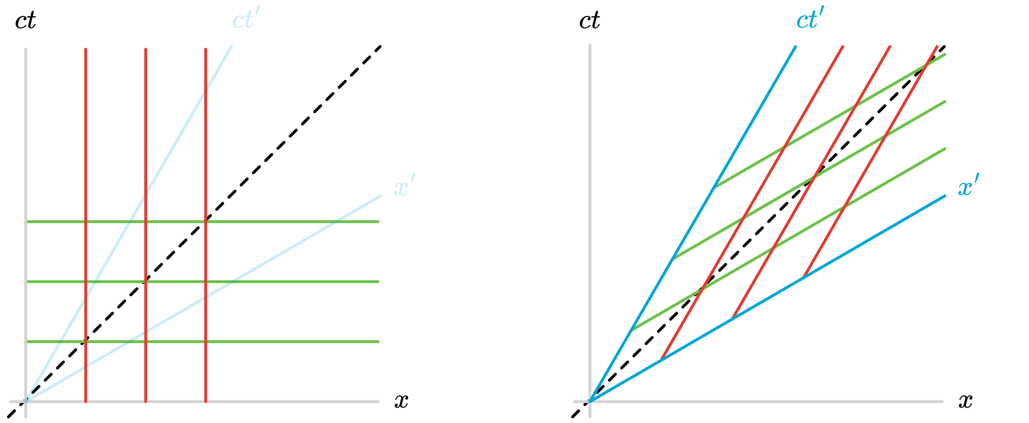


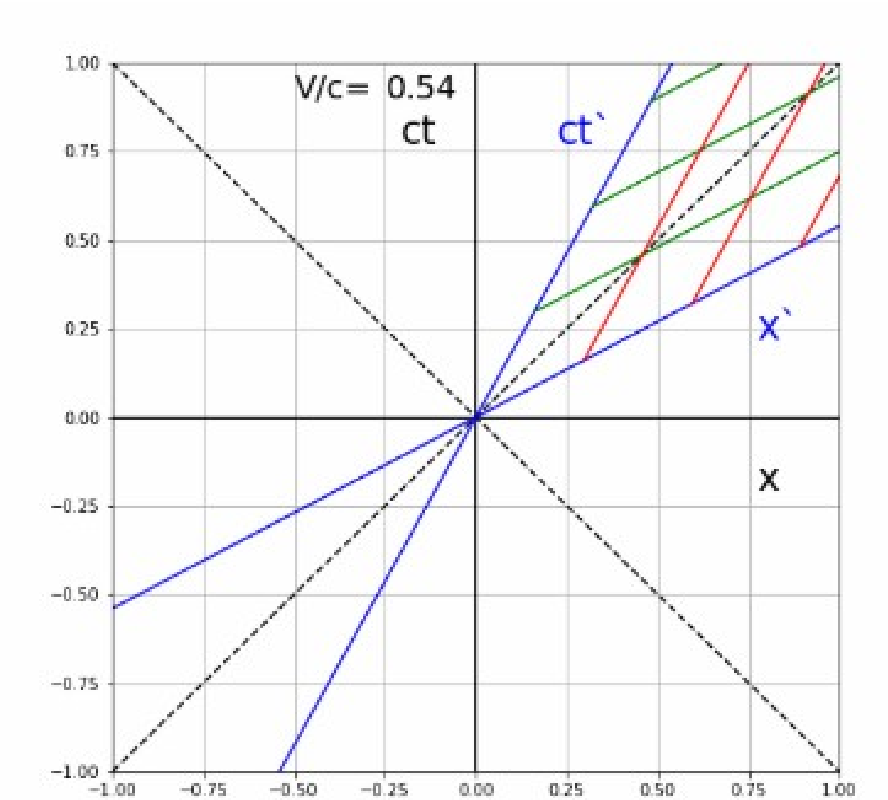
Figure 3.63: Left: lines of equal time (green) and equal space coordinate (red) in frame of reference S (left) and S' (right).

For the frame of reference S' that is only a bit different.

- Lines of constant time in S' are parallel to x'
- Lines of constant space coordinate in S' are parallel to ct'

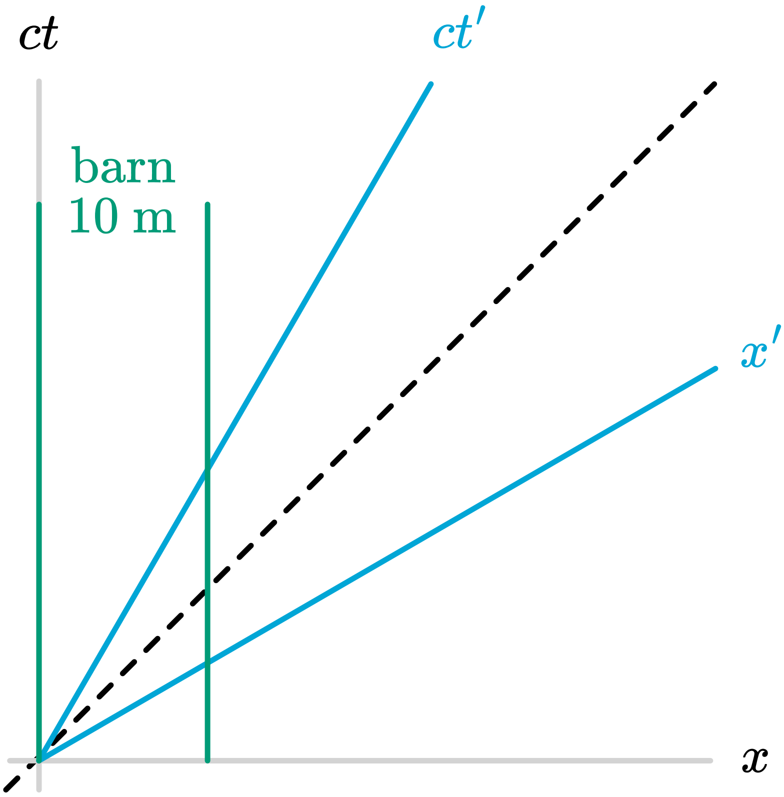
With this information in hand, we can investigate how events are transferred from S to S'. We can graphically do a LT without the explicit computation.

In the animation below, we see the effect of different values of V/c on the lines of constant ct' and x' as seen by S. For clarity, these are only drawn for $V/c \geq 0$

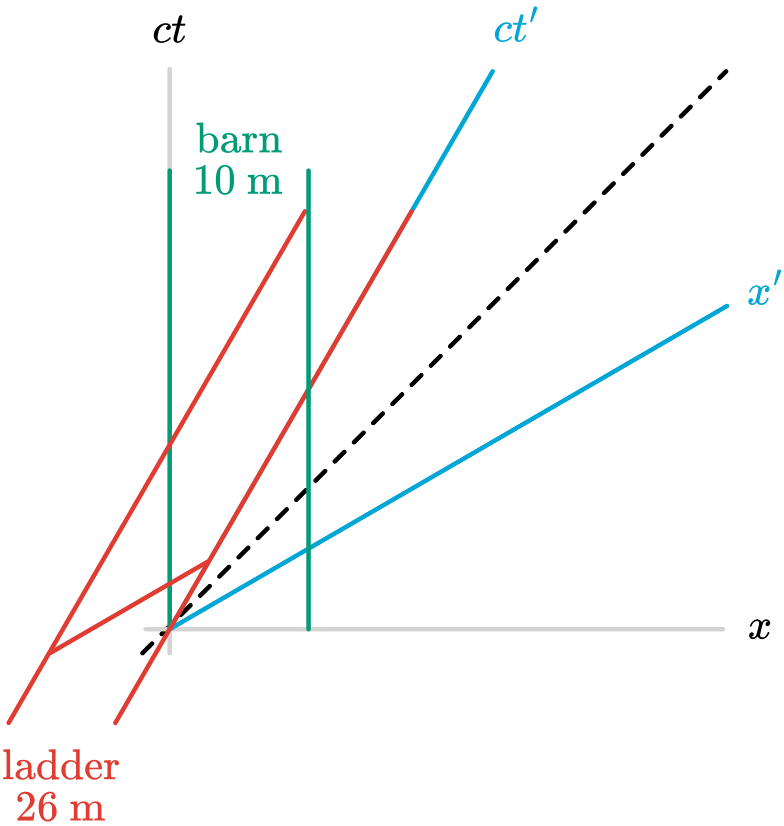


3.4.4.1 The ladder & barn revisited

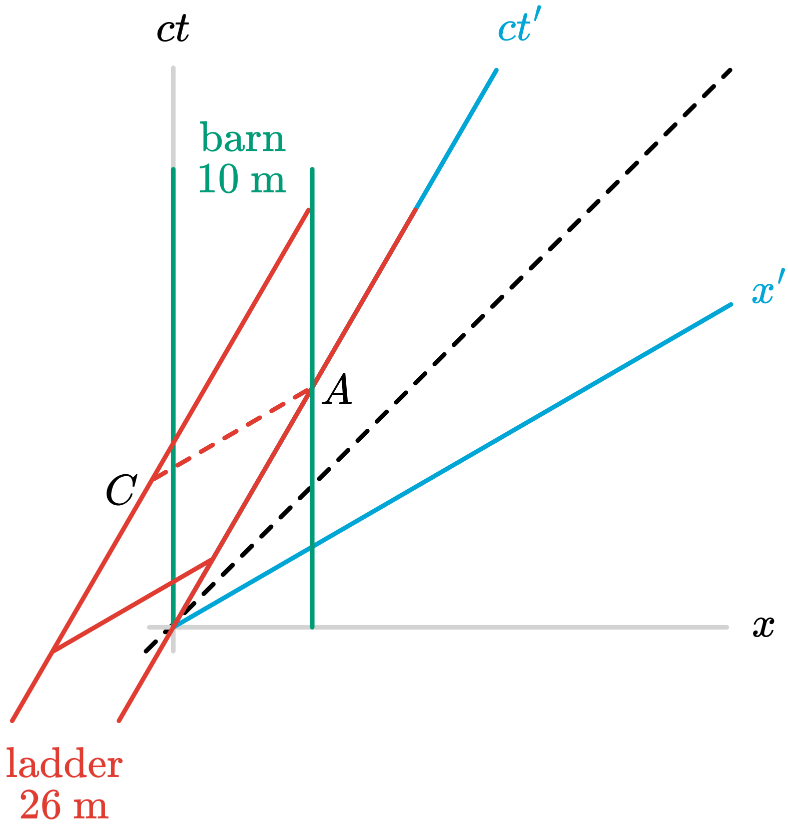
We will now take a look back at the ladder and barn paradox. We had a barn of 10 m wide and a ladder of 26 m long (both measured in their rest frame). The ladder was moving towards the barn with high velocity. We start by drawing the barn S (black) and ladder S' (blue) coordinate systems in the Minkowski diagram. Now we add the barn world line into the diagram (light blue) with 2 lines of constant space coordinate (parallel to ct) in S.



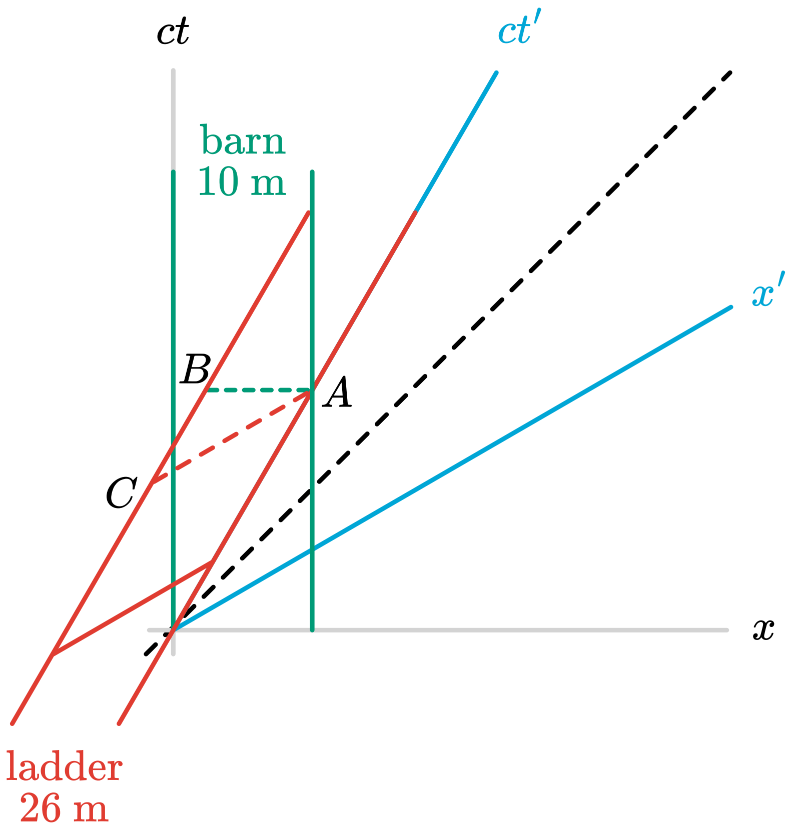
Now we can add the ladder to S'. It has rest length of 26 m and in the (x', ct') plane it is a world line of constant space coordinate, therefore parallel to ct'. The ladder itself is a line of constant time in ct' and therefore parallel to x'.



As the ladder moves (we move it parallel to x' between the world lines) it will eventually enter the barn and hit the right door of the barn (dashed red line). This event is indicated by the space time point A. For S' the other end of the ladder is then still outside the barn at space time point C. According to S' the ladder does not fit into the barn.



When the ladder hits the right door for S at space time point A, he makes a measurement of the ladder. To this end we draw a line of constant time (dashed light blue, parallel to x) until it intersects the world line of the ladder at space time point B. Observer S measures that the ladder fits into the barn.



From this diagram it is obvious that the events B and C are not the same, therefore it is not strange that S and S' disagree about the outcome of the measurement. Both are right! But they would not be able to agree that both doors shut at the same time, to capture the ladder.

3.4.4.2 The twin paradox

Let there be two twins, Alice and Bob. Bob leaves earth in a spaceship with relativistic speed $\vec{v}$, while Alice remains back home on earth. At some time Bob turns around, with $-\vec{v}$ and comes back to Alice. Based on time dilation Alice will argue that Bob is younger than she due to $\Delta T = \gamma \Delta T_0$. For the γ-factor it does not matter if Bob is moving away or approaching as it is quadratic in the velocity. For each year she ages, her brother only ages $1/\gamma$ years. Bob can argue that due to the principle of relativity, he is at rest and his twin sister is moving away and then coming back, therefore she will be younger than he - and we have a paradox.

This paradox has two issues:

1. The principle of relativity is not applicable as Bob must *turn around*. This requires acceleration of his frame and breaks the symmetry of the problem.
2. Bob will be younger than Alice, due to the relativity of simultaneity changing around the turning point. We can see this by looking at the Minkowski-diagram below. Just before Bob is turning around, his line of simultaneity is x', but just after turning around his line of simultaneity is x''. On the time line of Alice, Bob lines of simultaneity first is at point A, but then makes a jump around the turning point to B. Bob will be younger than Alice, by the length of this jump on her time line from A to B.

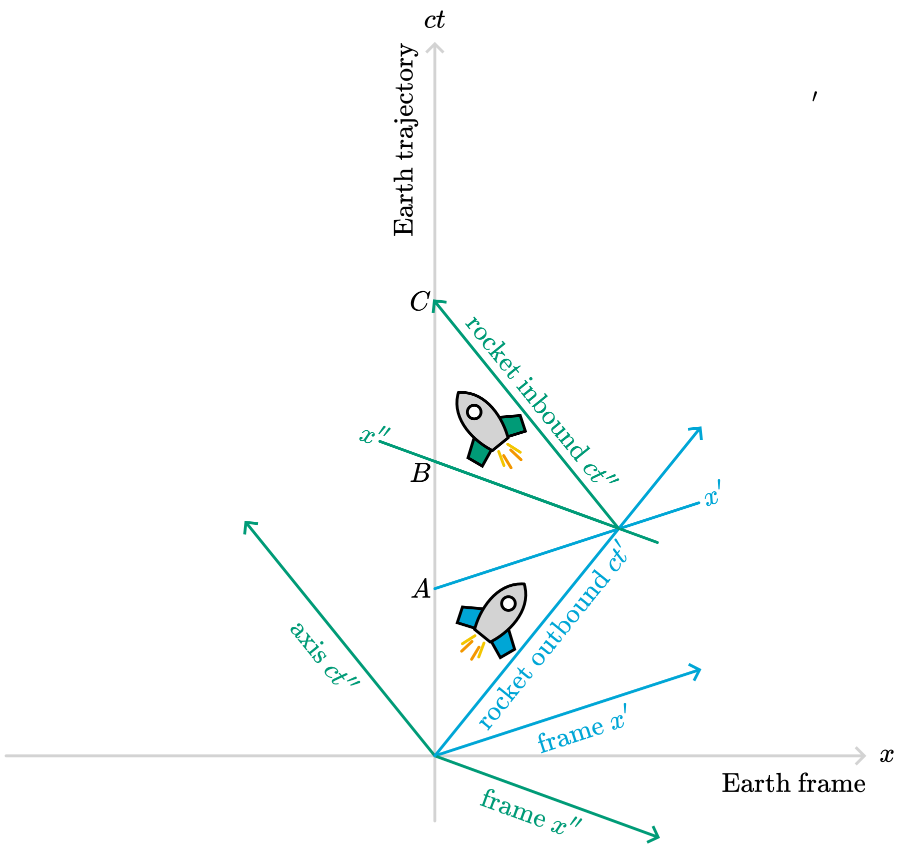


Figure 3.69: .

Extra: We symmetries the problem. Both Alice and Bob move in spaceships away from each other at the same but opposite speed, then turn around and meet again. Who is older now?

Answer

They are the same age. You can now reason with symmetry even though both are accelerated. You can also draw the Minkowski-diagram similar to the above and see that both make the same "jump" in the time, and thus are the same age.

Example: the rabbit and the turtle

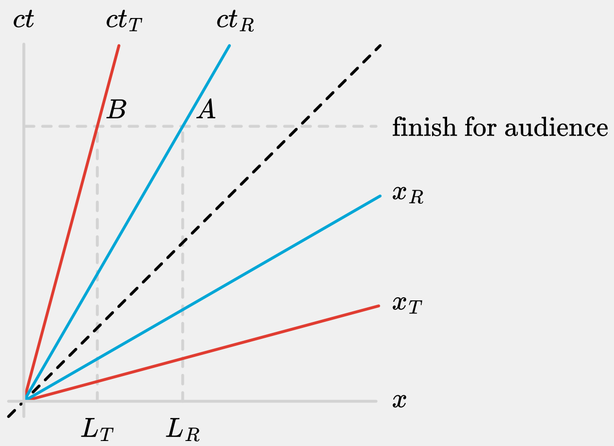
We consider the relativistic race between the well-known rabbit (R) with speed v_R and his buddy turtle (T) with speed $v_T < v_R$. Both turtle and rabbit are point particles. To give turtle a chance, it does not need to run as far as rabbit ($L_T < L_R$). The distances are chosen such that an observer at rest (the audience) records that R and T finish at the same time.

1. Draw a Minkowski-diagram of the situation described above.
2. Indicate the following events in space time.
    - R finishes in his frame (A)
    - T finishes in his frame (B)
    - In the frame of R, when he finishes, the event where T is then (C)
    - In the frame of T, when he finishes, the event where R is then (D)
3. Who has won according to R and who according to T. Do they agree?

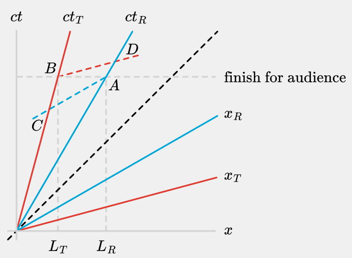
Solutions:

We start by drawing the audience frame with (ct, x) and an equal time line for the finish of R and T. From that we draw the coordinate system of R as (ct_R, x_R) and of T as (ct_T, x_T). As $v_T < v_R$, the coordinate system (ct_R, x_R) is closer to the light line. The length L_R and L_T follow as the intersections of ct_R and ct_T with the line of equal time for the audience.

These intersections are also directly the events A and B.



For the events C and D, we first draw from A a line of constant time for R (parallel to x_R) and then look at the intersection with the world line of T and mark it with C. The same for the event D. We draw a line parallel to x_T of constant time for T through B to see where R is when T finishes and mark it with D.



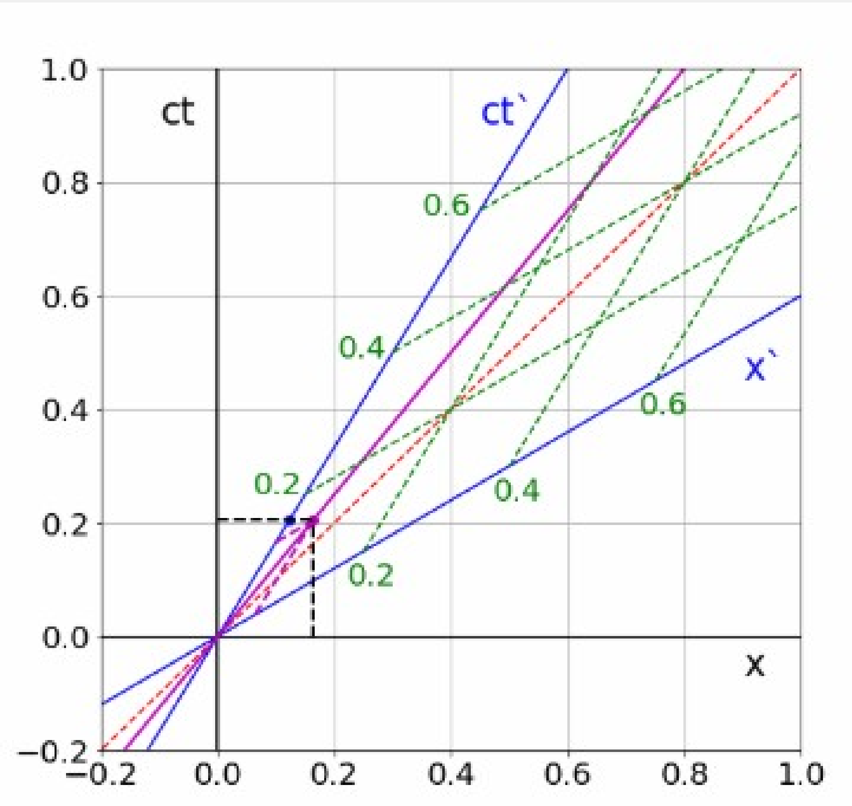
Both R and T agree that R has won, but the audience does of course not agree.

Example: moving particle

Consider a standard situations: S' moving at $V/c = 3/5$ with respect to S. Clocks are synchronized at $ct' = ct = 0$ when $x' = x = 0$.

According to S, a particle is moving with $U/c = 4/5$ over the x-axis. S describes the trajectory of the particle as $x_p(ct) = \frac{U}{c}ct$. In the animation below a Minkowski diagram is plotted as S would do. The motion of S' is made visible by the moving blue dot. Similarly, the pink dot shows the motion of the particle. The grey grid is giving coordinates according to S. The black dashed lines show the ct and x coordinate of the particle as S uses.

The green dashed lines is the grid of S' translated to the world of S. The pink dashed lines show the corresponding coordinates of the particle in the world of S': they intersect the ct' and x' axes at the position and time as S' would use. Notice that the clock of S' is indeed slow. Of course the x' coordinate of the particle stays relatively small: S' is 'chasing' the particle.



3.4.4.3 Lines of invariant distance

We have seen that the length interval ds^2 is a Lorentz invariant. Therefore we can use it to also indicate corresponding time and space units in a Minkowski diagram for two moving observers. If we fix ds^2 then the equation $ds^2 = c^2dt^2 - dx^2$ describes a hyperbola in (ct, x) of the Minkowski diagram.

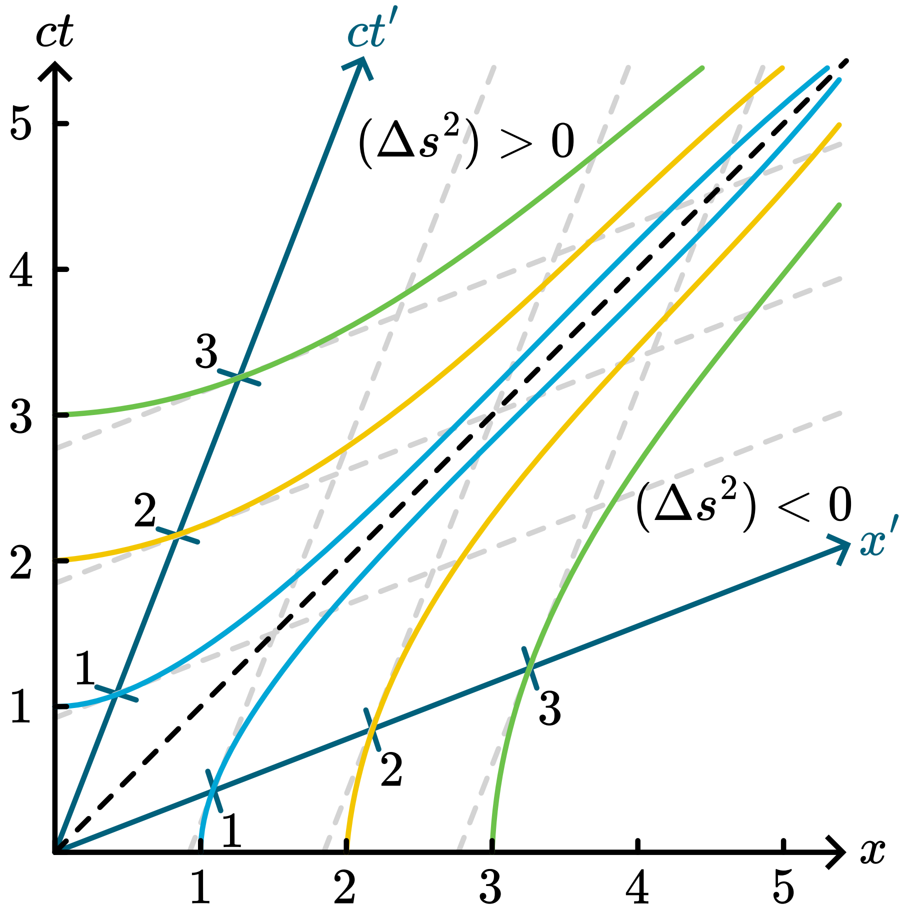


Figure 3.73: Image from Idema (2023)

For $ds^2 < 0$ we find the corresponding space units (the interval is space-like), and for $ds^2 > 0$ the corresponding time units (the interval is time-like. All hyperbola have the light line $ds^2 = 0$ as asymptotes.

Example: Circles are not circular??

We define a circle as the set of points (in a plane) that have the same distance to some given point M. We can easily extend this to three dimensions: that the circle becomes the surface of a sphere. If we stick to Eucledian spaces, we can do this for any dimension: a spherical surface in n-dimensional space, is the collection of points with the same distance to a given point M. Now the point has to be represented by n coordinates. But our measure of distance follows the same inner-product as we use in 2 and three dimensions:

let $\{M_i\} with i = 1..n$ be a point in n-dimensional space. Then all points $\{X_i\} with i = 1..n$ that obey the rule

$$\sum_{i=1}^{n} (X_i - M_i)(X_i - M_i) = R^2 \tag{3.82}$$

form a spherical surface with distance R to M. The above rule is actually the inner product of $\vec{X} - \vec{M}$: $\left(\vec{X} - \vec{M}\right) \cdot \left(\vec{X} - \vec{M}\right) = R^2$

Without loss of generality, we can chose the origin at M. That simplifies notation: $\vec{X} \cdot \vec{X} = R^2$ is now the surface of a sphere of radius R with center $\mathcal{O}$.

What if we leave our Eucledian space and go to the Minkowski space of special relativity? We still would define e circle as a set of point with the same distance to a given point. But now, our measure of distance is different. Let's again take the origin as the central point. Then, we are looking for the set of point $\{X^\mu\}_i$ such that $\vec{X} \cdot \vec{X} = R^2$. This means:

$$X^0X^0 - X^1X^1 - X^2X^2 - X^3X^3 = R^2 \tag{3.83}$$

Or, if we only consider ct, x:

$$c^2t^2 - x^2 = R^2 \tag{3.84}$$

These are the 'circles' in Minkowski ct, x-space. Of course, we would have the tendency to call them hyperbola, as they have the mathematical expression of a hyperbola. But in fact, the interpretation in Minkowsi space would be that of circles, that is the collection of points with the same distance to the origin.

Note, that $R = 0$ now does not reduce the set to a single point, but instead refers to the light lines. Second note: we do not have a problem here with negative distances. Thus if we take R to be a pure imaginary number, we will still get hyperbola, but just rotated by 90°.

3.4.5 LT as a rotation

This part is optional, but insightful.

You can think of the LT as a rotation of the 4 coordinates of Minkowski space time. Obviously it is not a "normal" rotation with a rotation matrix $R \in SO(n)$ as we encountered in change to polar coordinates.

The LT in matrix notation reads as follows with $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = V/c$.

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{3.85}$$

The matrix transfers the space time coordinates between two observers moving with V. From this it is clear that transferring between more than two observers $S \to S' \to S'' \to \ldots$ can be done easily by multiplying the respective Lorentz transformation matrices into one overall LT. This must be possible, of course, as the LT is a linear transformation in space time (ct, x).

From the matrix notation it is also clear that for rotations around "different axis", speeds in x, y, z direction, the order of change of frame matters as matrix multiplication does not commute.

In 3D normal space, distance is persevered under rotation with $R \in SO(n)$, in Minkowski space distance is preserved under Lorentz transformation which too is a rotation.

You can see the rotation clearer if we introduce the quantity *rapidity* α, which is defined as $\tanh\alpha \equiv \frac{V}{c}$ (a relativistic generalization of the modulus of the velocity. It goes from 0 for $v = 0$ to ∞ for $v = c$). We will not use the rapidity except here, however, it is used for relativistic velocity decompositions. With $\tanh\alpha = \frac{V}{c}$ we can write the Lorentz transformation as (using $\gamma = \frac{1}{\sqrt{1-\tanh^2\alpha}} = \cosh\alpha$ and $\gamma\beta = \frac{\tanh\alpha}{\sqrt{1-\tanh^2\alpha}} = \sinh\alpha$)

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{3.86}$$

Notice the similarity to the rotation with sine and cosine.

With that LT is a rotation in hyperbolic space with "angle" α (where α is the rapidity), we identify the matrix as $L(\alpha)$. That the hyperbolic functions appear should not be a surprise as they are equivalent to the sine and cosine for the circle, $(ct^2 + x^2 = 1)$, for the hyperbola $(ct^2 - x^2 = 1)$. Notice the relation to the inner products for standard and Minkowski space.

Minkowski made the sketch below to show that the Lorentz transformation is a rotation over a hyperbola not a circle as we were used to. The asymptotes of the hyperbola are given by the light lines.

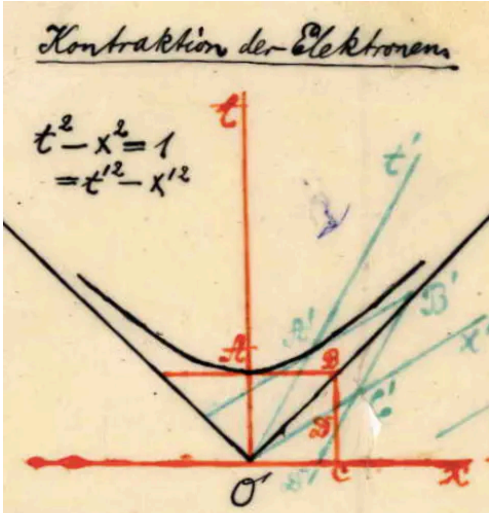


Figure 3.74: Drawing by Minkowski

The addition of velocities that we derived earlier is easy with this notation with rotations and rapidity $L(\alpha_1)L(\alpha_2) = L(\alpha_1 + \alpha_2)$. In terms of speeds this reads

$$\beta = \tanh(\alpha_1 + \alpha_2) = \frac{\tanh\alpha_1 + \tanh\alpha_2}{1 + \tanh\alpha_1 \tanh\alpha_2} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} \tag{3.87}$$

The addition of velocities is brought back to hyperbolic identities.

Exercise 3.75: Connected events 🌶️

Consider the following pairs of events and determine whether they are time like, space like or light like connected.

a. E1: $(ct_1, x_1) = (1, 0)$ and E2: $(ct_2, x_2) = (0, 1)$
b. E3: $(ct_3, x_3) = (1, 3)$ and E4: $(ct_4, x_4) = (-2, 1)$
c. E5: $(ct_5, x_5) = (1, 2)$ and E6: $(ct_6, x_6) = (3, 4)$

S' travels at $V/c = 12/13$ in the positive x-direction with respect to S. Their clocks are synchronized when their origins coincide.

d. Answer the same questions, but now from the perspective of S'.

Exercise 3.76: 🌶️

In the frame of S a laser is placed at $(x_1, y_1, z_1) = (4, 0, 0)$. A receiver is located at $(x_2, y_2, z_2) = (0, 3, 0)$. At $ct = 0$ the laser fires a light puls towards the receiver.

Find the events "pulse sent" and "pulse received" and determine the distance between the two events.

Secondly, an observer S' moves with $V/c = 4/5$ with respect to S. The velocity points in the positive x-direction. Both observers have their x resp. x' axis aligned and their clocks synchronized: $ct' = ct = 0$ when $x' = x = 0$.

Find the events for S' and show that the same distance is found between the two events.

3.4.6 Exercises, examples & solutions

Updated: 04 feb 2026

Exercise 3.77: 🌶️

Observer S' moves at a constant velocity of $V/c = 12/13$ with respect to S. They have aligned their axes and synchronized their clocks in the usual way.

Consider the two events $E1 : (ct_1, x_1) = (3, 3)$ and $E2 : (ct_2, x_2) = (4, 5)$

a. Compute the distance between the two events, Δs^2, according to S.
b. Compute the event coordinates according to S'.
c. Compute $\Delta s'^2$ and convince yourself that this is of course equal to Δs^2.

Exercise 3.78: 🌶️ 🌶️

Observer S' moves at a constant velocity of $V/c = 3/5$ with respect to S. They have aligned their axes and synchronized their clocks in the usually way. In the world of S' the following events happen:

E0: $(ct'_0, x'_0) = (0, 0)$ preparation is made to send a signal;
E1: $(ct'_1, x'_1) = (1, 0)$ a light signal is send over the positive x' axis;
E2: $(ct'_2, x'_2) = (2, 1)$ the signal is received;
E3: $(ct'_3, x'_3) = (3, 1)$ the signal is processed and a second one is emitted in the negative x' direction;
E4: $(ct'_4, x'_4) = (4, 0)$ the signal is received;
E5: $(ct'_5, x'_5) = (5, 0)$ the signal is processed.

Find the corresponding (ct, x) coordinates according to S. Draw the events in two diagrams. The first one has both ct and ct' as the vertical axis and x and x' as the horizontal axis. The second on is a Minkowski diagram from the perspective of S.

Exercise 3.79: 🌶️ 🌶️

A spaceship, with S' on board, is moving at $V/c = 3/5$ with respect to Mission Control (where S is located) on earth. Both S and S' have aligned their axes and synchronized their clocks in the usual way.

Mission control receives at $t = 1.7$ ls images from the impact of a meteorite on the moon. The distance from Mission Control to the moon is 1.2ls (according to S). This event E1. Event E2 is the impact itself (that is where and when of the impact), Event 3 is the receiving of images of the impact by S'. Of course, images travel in space at the speed of light.

a. Transform the events to S' using a Lorentz Transformation. b. Find the position of S' at the time of the three events according to S. This provides additional events. c. Make a (ct, x) diagram in which you plot all the above events. Draw the the world line of S' in the diagram. d. Do the same but now for S'. e. Make a Minkowski diagram (from the perspective of S) and draw the grid-lines of S' for the events E1 and E2.

Solution 3.80: Solution to Exercise 1

a. E1: $(ct_1, x_1) = (1, 0)$ and E2: $(ct_2, x_2) = (0, 1)$

$$\rightarrow \Delta s_{12}^2 = (1-0)^2 - (0-1)^2 = 0 lightlike \tag{3.88}$$

b. E3: $(ct_3, x_3) = (1, 3)$ and E4: $(ct_4, x_4) = (-2, 1)$

$$\rightarrow \Delta s_{34}^2 = (1+2)^2 - (3-1)^2 = 5 timelike \tag{3.89}$$

c. E5: $(ct_5, x_5) = (1, 2)$ and E6: $(ct_6, x_6) = (3, 4)$

$$\rightarrow \Delta s_{56}^2 = (1-3)^2 - (2-4)^2 = 0 lightlike \tag{3.90}$$

d. Transform to S': $V/c = 12/13 \rightarrow \gamma = 13/5$

$$\begin{aligned} ct' &= \gamma\left(ct - \frac{V}{c}x\right) \\ x &= \gamma\left(x - \frac{V}{c}ct\right) \end{aligned} \tag{3.91}$$

E1: $(ct'_1, x'_1) = (13/5, -12/5)$ and E2: $(ct_2, x_2) = (-12/5, 13/5)$

$$\rightarrow \Delta s'^2_{12} = (13/5 + 12/5)^2 - (-12/5 - 13/5)^2 = 0 lightlike \tag{3.92}$$

E3: $(ct'_3, x'_3) = (-23/5, 27/5)$ and E4: $(ct_4, x_4) = (-38/5, 37/5)$

$$\begin{aligned} \rightarrow \Delta s'^2_{34} &= (-23/5 + 38/5)^2 - (27/5 - 37/5)^2 \\ &= 225/25 - 100/25 = 5 timelike \end{aligned} \tag{3.93}$$

E5: $(ct'_5, x'_5) = (-11/5, 14/5)$ and E6: $(ct'_6, x'_6) = (-9/5, 16/5)$

$$\rightarrow \Delta s'^2_{56} = (-11/5 + 9/5)^2 - (14/5 - 16/5)^2 = 0 lightlike \tag{3.94}$$

Of course, for all cases we find $\Delta s'^2 = \Delta s^2$: distance defined according to our Minkowski inproduct is a Lorentz invariant, i.e. the same for all inertial observers.

Solution 3.81: Solution to Exercise 2

For S:

$$E1 : (ct_1, x_1, y_1, z_1) = (0, 4, 0, 0) \tag{3.95}$$

$$E2 : (ct_2, x_2, y_2, z_2) = (5, 0, 3, 0) \tag{3.96}$$

$$\delta s_{12}^2 = (0-5)^2 - (4-0)^2 - (0-3)^2 - (0-0)^2 = 0 \tag{3.97}$$

light-like of course, as we deal with a light pulse.

For S': LT with $V/c = 4/5 \to \gamma = 5/3$

$$\begin{aligned} ct' &= \frac{5}{3}\left(ct - \frac{4}{5}x\right) \\ x' &= \frac{5}{3}\left(x - \frac{4}{5}ct\right) \\ y' &= y \\ z' &= z \end{aligned} \tag{3.98}$$

Thus:

$$E1 : (ct'_1, x'_1, y'_1, z'_1) = (-16/3, 20/3, 0, 0) \tag{3.99}$$

$$E2 : (ct'_2, x'_2, y'_2, z'_2) = (25/3, -20/3, 3, 0) \tag{3.100}$$

$$\begin{aligned} \delta s_{12}'^2 &= (-16/3 - 25/3)^2 - (20/3 + 20/3)^2 - (0-3)^2 - (0-0)^2 \\ &= \frac{41^2}{9} - \frac{40^2}{9} - \frac{81}{9} = 0 \end{aligned} \tag{3.101}$$

Solution 3.82: Solution to Exercise 3

We start with writing down the LT. As $V/c = 12/13$ we have $\gamma = 13/5$. Thus, for this case LT reads as:

$$\begin{aligned} ct' &= \frac{13}{5}\left(ct - \frac{12}{13}x\right) \\ x' &= \frac{13}{5}\left(x - \frac{12}{13}ct\right) \end{aligned} \tag{3.102}$$

a.

$$\begin{aligned} \Delta s^2 &\equiv (ct_2 - ct_1)^2 - (x_2 - x_1)^2 \\ &= (4-3)^2 - (5-3)^2 \\ &= -3 \end{aligned} \tag{3.103}$$

b. event E1:

$$\begin{aligned} ct'_1 &= \frac{13}{5}\left(3 - \frac{12}{13}3\right) = \frac{3}{5} \\ x'_1 &= \frac{13}{5}\left(3 - \frac{12}{13}3\right) = \frac{3}{5} \end{aligned} \tag{3.104}$$

event E2:

$$\begin{aligned} ct'_2 &= \frac{13}{5}\left(4 - \frac{12}{13}5\right) = -\frac{8}{5} \\ x'_2 &= \frac{13}{5}\left(5 - \frac{12}{13}4\right) = \frac{17}{5} \end{aligned} \tag{3.105}$$

c.

$$\begin{aligned} \Delta s'^2 &\equiv (ct'_2 - ct'_1)^2 - (x'_2 - x'_1)^2 \\ &= \left(-\frac{8}{5} - \frac{3}{5}\right)^2 - \left(\frac{17}{5} - \frac{3}{5}\right)^2 \\ &= \frac{121}{25} - \frac{196}{25} = -3 \end{aligned} \tag{3.106}$$

Solution 3.83: Solution to Exercise 4

Lorentz Transformation

$$ct = \gamma\left(ct' + \frac{V}{c}x'\right)$$
$$x = \gamma\left(x' + \frac{V}{c}ct'\right) \tag{3.107}$$
$$with \frac{V}{c} = \frac{3}{5} and \gamma = \frac{5}{4}$$

This gives:

E0: $(ct_0, x_0) = (0, 0)$
E1: $(ct_1, x_1) = (5/4, 3/4)$
E2: $(ct_2, x_2) = (13/4, 11/4)$
E3: $(ct_3, x_3) = (9/2, 7/2)$
E4: $(ct_4, x_4) = (5, 3)$
E5: $(ct_5, x_5) = (25/4, 15/4)$

This gives the two required plots.

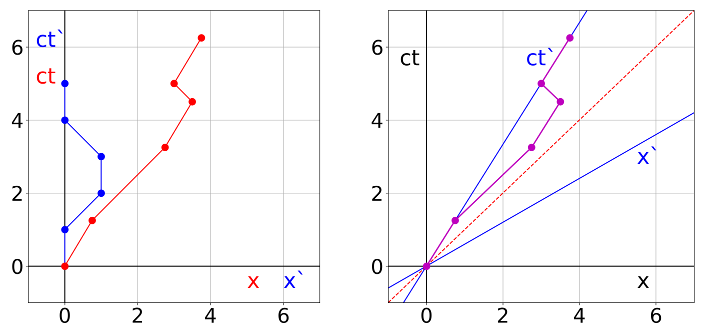


Figure 3.84: left: S and S' in the same diagram; right: Minkowski diagram.

Solution 3.85: Solution to Exercise 5

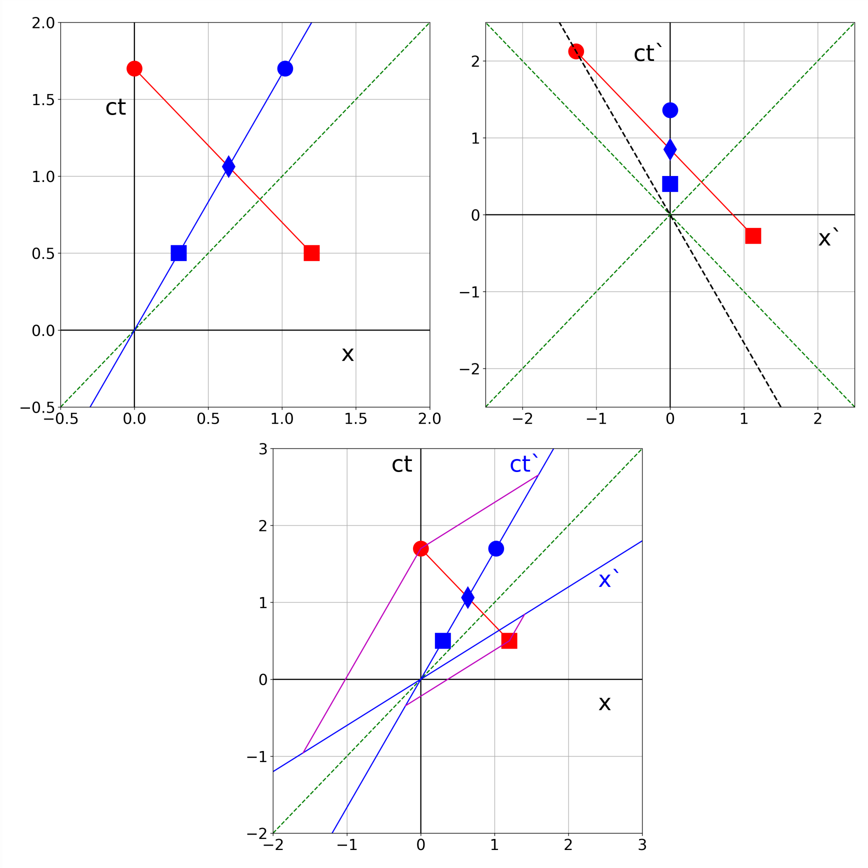


Figure 3.86: top left: S , top right: S', bottom: Minkowski diagram.
red: square - comet hits moon, diamond - photon registered by spaceship, circle - photon detected by earth
blue: corresponding position of S' according to S and its Lorentz Transformation for S'
Minkowski diagram: pink lines show the intersection with the ct' and x' axes, i.e. the coordinates according to S'

3.4.6.1 Exercises

3.4.6.2 Answers

3.5 4-Momentum & E=mc²

Updated: 04 feb 2026

3.5.1 Proper time

We have seen that in Special Relativity events are described by four coordinates: (ct, x, y, z). Moreover, distance is measured via a inner product $A^\mu \cdot B^\mu = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3$. That opens the question: what about other quantities that we use in mechanics?

If position is $X^\mu = (ct, x, y, z)$ then what is velocity? Is $v^\mu \equiv \frac{dX^\mu}{dt}$ a good choice? It is what we are used to: velocity is change in position over time. However, we need to be careful. We require that our quantities are four-vectors, i.e. they transform according to the Lorentz Transformation. And the length, i.e. the inner product with itself, is the same for all inertial observes.

However, in our our first choice of the definition, we take the derivative with respect to time. But time is not the same for different observers!

We do know that the distance ds^2 is LT invariant, as is c^2, therefore we can combine both into another invariant - of time

$$d\tau^2 \equiv \frac{ds^2}{c^2} \tag{3.108}$$

If we spell out ds^2 we can write

$$d\tau^2 = \frac{ds^2}{c^2} = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) \tag{3.109}$$

$d\tau$ is called *proper time* or *Eigenzeit* because for the rest frame S' we have $(dx' = dy' = dz' = 0)$ and thus

$$d\tau^2 = dt'^2 \tag{3.110}$$

We associate to a moving particle the 3-velocity $\vec{u} = \left(u_x, u_y, u_z\right) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$. This is the velocity that we normally use: it is distance as measured in our frame of reference over time as we see on our clocks. We can relate the proper time $d\tau$ to the frame/coordinate time dt:

$$\begin{aligned} d\tau^2 &= dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) \\ &= dt^2 \left[1 - \frac{1}{c^2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right)\right] \end{aligned} \tag{3.111}$$

Here we use the magnitude of the 3-velocity u. In other words

$$\frac{d\tau^2}{dt^2} = 1 - \frac{u^2}{c^2} \Rightarrow dt = \gamma(u) d\tau \tag{3.112}$$

The proper time interval relates to the frame time via the γ-factor for the velocity u.

3.5.2 4-velocity

Now we can tackle the 4-velocity. In order to make any sense we must define a velocity whose length is an invariant. Furthermore, velocity must be something like displacement over time interval. For the displacement the obvious choice is: dX^μ, i.e. a particle has moved from X^μ to $X^\mu + dX^\mu$. The displacement dX^μ transforms, of course, via the Lorentz Transformation. Moreover, its length is a Lorentz Invariant. In order to arrive at an adequate velocity, we must thus divide the displacement by a time interval that is also a Lorentz Invariant. Luckily, we have just seen that proper time is a Lorentz Invariant.

Therefore the 4-velocity $\vec{U}$ is

$$U^\mu \equiv \frac{dX^\mu}{d\tau} \tag{3.113}$$

where the derivative of the 4-position vector is taken with respect to the proper time τ. We obtain the relation to the 3-velocity $\vec{u}$ just from filling in $d\tau = dt/\gamma(u)$

$$U^\mu = \gamma(u)\left(\frac{dct}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (\gamma(u)c, \gamma(u)\vec{u}) \tag{3.114}$$

4-velocity transfers between frames moving with speed V as given by the Lorentz transformation as $\vec{U}$ is a 4-vector.

3.5.2.1 Be careful with 4-vector interpretation

We compute the inner product of $\vec{U}$ with itself $U^2 = \gamma^2(u)(c^2 - u^2)$. That is a LT invariant of course. Therefore we can choose the frame such that $u = 0$, or in other words $U^2 = c^2$. The 4-velocity length is constant! That is not intuitive at all. Even stranger as the vector has constant length, it follows that the 4-velocity is always perpendicular to the 4-acceleration.

$$\frac{d}{d\tau}U^2 = 2\vec{U} \cdot \frac{d}{d\tau}\vec{U} = 0 \tag{3.115}$$

The counter intuitive stuff happens of course due to the pseudo-Euclidean metric.

3.5.2.2 Revisit 3-velocity transformation

Earlier we transformed the velocity u of a particle in S to S' which was moving with V. This was quite complicated and the formula is difficult to remember. However, there is no need to remember the formula, you can always derive it from the transformation of the 4-velocity.

The 4-velocity of a particle moving at velocity $\vec{u} = \left(u_x, u_y, u_z\right)$ according to observer S is as we have seen:

$$\vec{U} = (\gamma(u)c, \gamma(u)\vec{u}) \tag{3.116}$$

.

Note that the gamma-factor here is a property of the particle, that is: it is the gamma-factor as S would calculated based on the particle velocity that S observes: $\gamma(u) \equiv \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$. It has nothing to do with the gamma-factor that S and S' use in their Lorentz Transformation to exchange information.

As we can write down the LT of a 4-vector between S and S'.

$\vec{U}$ is a four-vector it transforms from S to S' via the LT:

$$\begin{array}{rcl} \gamma(u')c & = & \gamma(V)\left(\gamma(u)c - \frac{V}{c}\gamma(u)u_x\right) \\ \gamma(u'){u'}_{x'} & = & \gamma(V)\left(\gamma(u)u_x - \frac{V}{c}\gamma(u)c\right) \\ \gamma(u'){u'}_{y'} & = & \gamma(u)u_y \\ \gamma(u'){u'}_{z'} & = & \gamma(u)u_z \end{array} \tag{3.117}$$

Note: here $\gamma(V)$ is the gamma-factor corresponding to the relative velocity V between S and S'. And $\gamma(V)$ has nothing to do with $\gamma(u)$. On the other hand $\gamma(u')$ is the gamma-factor that S' uses for the four-velocity of the particle that has, according to S' three-velocity $\vec{u}'$.

If we now divide the second of these equations by the first we obtain

$$\frac{{u'}_{x'}}{c} = \frac{\frac{u_x}{c} - \frac{V}{c}}{1 - \frac{Vu_x}{c^2}} \tag{3.118}$$

and if we divide the third of these equations by the first we obtain

$$\frac{{u'}_{y'}}{c} = \frac{\frac{u_y}{c}}{\gamma(V)\left(1 - \frac{Vu_x}{c^2}\right)} \tag{3.119}$$

Just what we have derived before, but now in a way that you can always do this on the spot if you know the definition of the 4-velocity and the LT of a 4-vector.

3.5.3 4-momentum

If we postulate that the mass m is LT invariant we can define the 4-momentum simply by

$$\vec{P} = m\vec{U} = (m\gamma(u)c, m\gamma(u)\vec{u}) \equiv (P^0, \vec{p}) \tag{3.120}$$

with the 3-momentum $\vec{p} = m\gamma(u)\vec{u} = m\frac{d\vec{x}}{d\tau}$.

> **mass is an LT invariant**
>
> The mass m *does not* change as a function of velocity $\vec{u}$. You still sometimes see $\tilde{m} \equiv \gamma(u)m$ and with this $\vec{P} = (\tilde{m}c, \tilde{m}\vec{u})$. That is not practical as it mixes kinetic energy with inertial mass.

3.5.3.1 Conservation of 4-momentum

For collisions now the total 4-momentum is conserved (per component)

$$\sum_{i,before} \vec{P}_i = \sum_{j,after} \vec{P}_j \tag{3.121}$$

If the total momentum is conserved than this must hold for the components $(m\gamma(u)c, \vec{p})$.

Note, that we did not write "mass is conserved". We postulate that it is a LT invariant, that is: it is the same for all inertial observers. But that does not imply that for collisions the mass should equal before and after the collision.

3.5.4 E=mc²

The most famous equation in physics.

We will derive it by looking at N2 in its relativistic form.

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\gamma(u)\vec{u}) = m\frac{d\vec{u}}{d\tau} \tag{3.122}$$

Kinetic energy was defined as work done on a mass. We again start from that and fill in N2 and take it step by step

$$\begin{aligned} \Delta E_{kin} &= \int_1^2 \vec{F} \cdot d\vec{r} = \int_1^2 \vec{F} \cdot \vec{u}dt \\ &= \int_1^2 \frac{d}{dt}(m\gamma(u)\vec{u}) \cdot \vec{u}dt \\ &= m\int_0^{\tilde{u}} \vec{u} \cdot d\gamma(u)\vec{u} \end{aligned} \tag{3.123}$$

This integration is more difficult than what we had before as the $\gamma(u)$ factor appears additional in the differential (for small velocities we have $\gamma(u) = 1$ and we just get $\frac{1}{2}mu^2$ as before). So, we will first give the outcome and the look into the details:

$$\begin{aligned} \Delta E_{kin} &= m\int_0^{\tilde{u}} \vec{u} \cdot d\gamma(u)\vec{u} \\ &= mc^2\ (\ \gamma(\tilde{u}) - 1\ ) \end{aligned} \tag{3.124}$$

How did we do this? We apply integration by parts. below is the full derivation. If you have difficulties following the math: don't worry, you will get this in your Calculus courses. For now, if you struggles with it: just skip the derivation and remember the outcome given above.

Warning

not in pdf yet

Integration by parts

Easy to remember integration by parts formula, from the product rule

$$\begin{aligned} (fg)' &= f'g + fg' \\ \Rightarrow \int (fg)' &= \int f'g + \int fg' \\ \int f'g &= [fg] - \int fg' \end{aligned} \tag{3.125}$$

In the derivation of the kinetic energy we used $f' = d\gamma(u)\vec{u}$ and $g = \vec{u}$.

If we now inspect the limiting cases for the velocity

$$\Delta E_{kin} = mc^2(\gamma(u) - 1) \tag{3.126}$$

- particle at rest: $u = 0 \Rightarrow \gamma(u) = 1 \Rightarrow \Delta E_{kin} = 0$
- small velocity $\frac{u}{c} \ll 1 \Rightarrow \gamma(u) = 1 + \frac{1}{2}\frac{u^2}{c^2} + \texttt{cal}\ O\left(\frac{u^4}{c^4}\right) \Rightarrow \Delta E_{kin} = \frac{1}{2}mu^2$

The limiting cases work out. Very reassuring.

4-momentum

Now that we have the kinetic energy, $mc^2(\gamma(u) - 1)$, we can compare that with the zeroth component of the 4-Momentum: $P^0 = m\gamma(u)c$. As we expected, the energy has a unit c extra: momentum and energy differ unit wise by m/s. But we also see that the kinetic energy (even if we divide it by c, i.e. use E_{kin}/c as the zeroth component) it is still not ok. The term $\gamma(u) - 1$ should have been $\gamma(u)$.

To cure this, we can add a constant (provide it is LT invariant) to the kinetic energy $E = E_{kin} + mc^2 = m\gamma(u)c^2$. Adding constants to the energy/potential is always allowed as only the change of it is physically relevant (or the relative energies).

We obtain

$$E = m\gamma(u)c^2 \tag{3.127}$$

or in the rest frame $(u = 0 \Rightarrow \gamma(u) = 1)$

$$E = mc^2 \tag{3.128}$$

With this energy $E = m\gamma(u)c^2$ we can define the 4-momentum as follows (we had $\vec{P} = (m\gamma(u)c, \vec{p})$)

$$\vec{P} = \left(\frac{E}{c}, \vec{p}\right) \tag{3.129}$$

4-momentum with a different energy?

With a different energy (addition of another constant to E_{kin} than what we did above) the length of the 4-momentum would not be LT invariant and $\vec{P}$ not a 4-vector. If we would have used $E = mc^2(\gamma - 1)$ then P^2 would not be LT invariant. You see this by computing $P^2 = \frac{E_{kin}^2}{c^2} - p^2c^2 = m^2c^2(2 - 2\gamma)$.

And we have finally derived *the* most famous equation in physics. We will use, however, $E = m\gamma(u)c^2$ most of the time as we are not always in the rest frame. The equation says essentially that mass is the same as energy. They are different manifestations of the same thing. A particle has energy in itself at rest without being in any potential.

Note

> As gravitation acts on mass, it should also act on energy if they are the same! This is indeed the case, also photons, massless particles, feel gravity. More about that in Einstein's theory of general relativity.

3.5.4.1 Mass in units of energy

The mass of an electron $m_e = 9.13 \cdot 10^{-31}\text{kg}$ is often given as $512 \ \text{keV}$, [kilo electron Volts]. Mass of all elementary particles is given actually in units of eV.

One electron volt is

$$1 \ \text{eV} = 1.6 \cdot 10^{-19}\text{C} \cdot 1\text{V} = 1.6 \cdot 10^{-19}\text{J} \tag{3.130}$$

The conversion to mass via $E = mc^2$

$$m_e c^2 = 8.2 \cdot 10^{-14}\text{J} = \frac{8.2 \cdot 10^{-14}}{1.6 \cdot 10^{-19}} = 512 \ \text{keV} \tag{3.131}$$

3.5.4.2 The fame

The origin of the fame is probably twofold.

- Firstly, mass is no longer conversed as was a central pillar in Newton's mechanics. It can be converted. This was shocking for *physicists only*.
- Secondly, when mass is actually converted into energy e.g. in a nuclear fission bomb or inside the sun with nuclear fusion, the effect is immense. The drop of the two nuclear bombs (little boy and fat man) on Hiroshima and Nagasaki made the equation inglorious world-known; life changing for *all people*.
- Einstein's rock star status helped certainly quite a bit.

3.5.5 Energy-momentum relation

The 4-momentum is, of course, a 4-vector and therefore P^2 is LT invariant. Let us have a look at the outcome with $\vec{P} = \left(\frac{E}{c}, \vec{p}\right)$

$$\begin{aligned} P^2 = \tfrac{E^2}{c^2} - p^2 &= \quad m^2\gamma^2(u)c^2 - m^2\gamma^2(u)u^2 \\ &= m^2\gamma^2(u)c^2\left(1 - \tfrac{u^2}{c^2}\right) = m^2c^2 \\ \Rightarrow E^2 - p^2c^2 &= \qquad\qquad m^2c^4 \end{aligned} \tag{3.132}$$

Indeed, we find that P^2 is LT invariant as m and c are LT invariants. Rearranging the equation, we obtain

$$E^2 = \left(mc^2\right)^2 + (pc)^2 \tag{3.133}$$

This converts back to $E = mc^2$ in the rest frame.

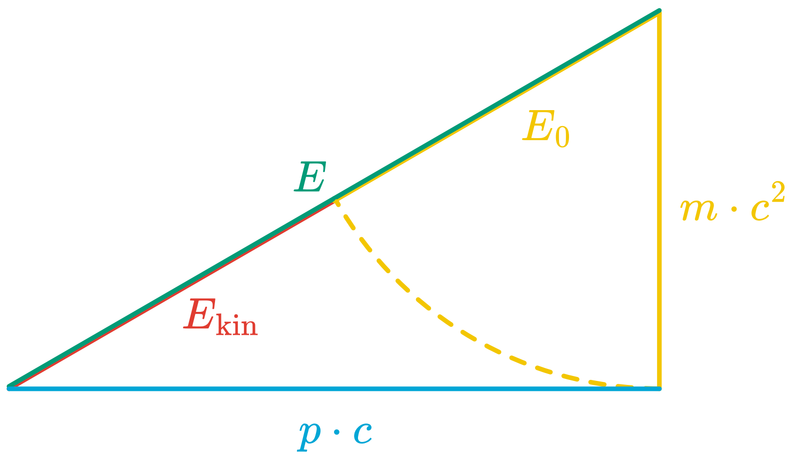


Figure 3.87: Einstein triangle.

You can visualize the energy momentum relation with the Einstein triangle shown here, as the relation has the form of $c^2 = a^2 + b^2$. With the kinetic energy as $E_{kin} = mc^2(\gamma(u) - 1)$. $E = E_0 + E_{kin} \equiv mc^2 + E_{kin}$.

3.5.5.1 LT invariance of P²

Above we found a very useful, but bit hidden relation in the derivation

$$P^2 = m^2c^2 \tag{3.134}$$

This is of course LT invariant, as m and c are LT invariants (and the momentum is a 4-vector), but more importantly we can use this for computations of relativistic collisions. By the conservation of 4-momentum we can of course compute all collisions by equating the 4 components of the momentum before and after the collision. It is often, however, mathematically easier to write down the conservation of momentum and then square it. Because you can write down $P^2 = m^2c^2$ directly, this saves often computations.

3.5.6 Photons

For photons we have the energy given by $E = \hbar\omega$ and the momentum as $p = \frac{\hbar\omega}{c}$. The 4-momentum of a photon is

$$\vec{P} = P^\mu = \left(\frac{E}{c}, \vec{p}\right) = \left(\frac{\hbar\omega}{c}, \frac{\hbar\omega}{c}\right)\left(\frac{h\nu}{c}, \frac{h\nu}{c}\right) \tag{3.135}$$

It is directly clear that for photons the LT invariant $P^2 = 0$.

We could substitute the photon 4-momentum into the energy-momentum relation, we find

$$E^2 = (pc)^2 + \left(mc^2\right)^2 \Rightarrow m = 0 \tag{3.136}$$

This seems to confirm that photons do not have mass. But we need to be careful: photons do not have a 4-momentum of the form $P^\mu = (m\gamma c, m\gamma u)$. They can't: (1) their velocity is always c, which would lead to ∞ for their $\gamma(c)$, (2) with a mass $m = 0$ we multiply γc by zero. Together, this would gives us $0 \times \infty$ which is not defined in a unique way.

Thus: photons do not have mass. Do not get confused with $E = mc^2$.

3.5.6.1 Rest frame of a photon?

Does a photon have a rest frame? It travels with the speed of light c (obviously) in all frames.

The answers is no and we give three good arguments.

- A rest frame implies that in this frame the object is at rest. But for a photon, traveling at c, which is LT invariant, there is no frame at which it is at rest, but only frames with $v = c$.
- The proper time of a photon is $d\tau^2 = dt^2 - \frac{1}{c^2}d\vec{x}^2$ but this is always equal to 0! A photon does not experience the passage of time, therefore it is reasonable to state that do not have a rest frame.
- In the hypothetical rest frame for a photon there would be no electro-magnetic radiation/interaction possible. In this frame e.g. the interaction between electrons would be zero.

3.5.6.2 Doppler revisited

In chapter 14 we discussed the Doppler effect from a relativistic point of view. With the concept of 4-momentum it is easy to derive the Doppler shift of photons as observed in different frames of reference. We take the usual LT between S' and S. In S' a photon is moving along the x'-direction. It has frequency f'. Its 4-momentum is

$${P'}^\mu_{photon} = \left(\frac{hf'}{c}, \pm\frac{hf'}{c}\right) \tag{3.137}$$

The $\pm$-sign indicates the direction of the photon: + for moving in the positive x'-direction, - for moving in the negative x'-direction.

Using the Lorentz Transformation, we can easily transform the 4-momentum to the frame of S:

$$\begin{aligned} \frac{hf}{c} &= \gamma\left(\frac{hf'}{c} + \frac{V}{c}\frac{\pm hf'}{c}\right) = \gamma\left(1 \pm \frac{V}{c}\right)\frac{hf'}{c} \Rightarrow \\ \frac{f}{f'} &= \frac{1 \pm V}{\sqrt{1 - V^2}} \end{aligned} \tag{3.138}$$

Note that we didn't use the transformation of P'^1_{photon} as this will give the same result.

3.5.7 Speed of light as limiting velocity

The γ factor increases strongly if the speed approaches the speed of light $u/c \to 1$ as can be seen in this plot

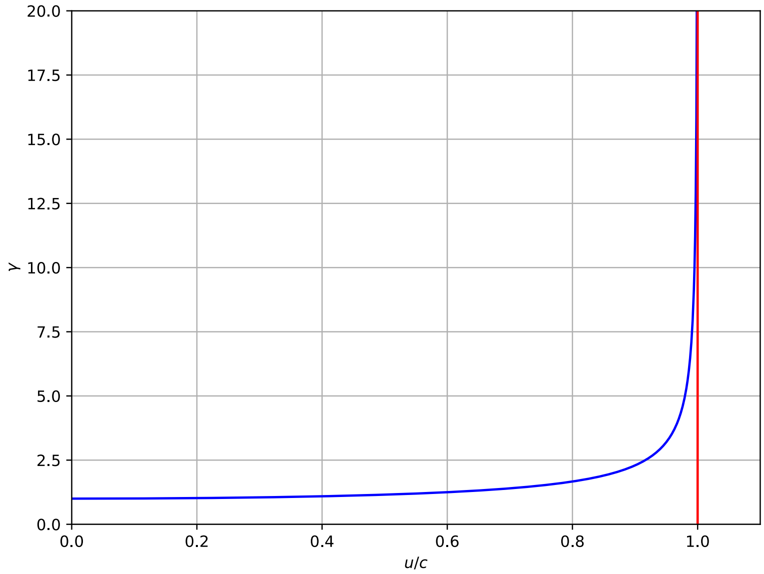


Figure 3.88: The γ factor increases strongly if the speed approaches the speed of light $u/c \to 1$

For a massive particle this has strong consequences. In the limit $u \to c$ the factor goes towards infinity. If we consider that the kinetic energy is $E = m(\gamma(u) - 1)c^2$, the amount of work done to increase the speed increases with γ. Therefore no massive particle can move with the speed of light (or faster) as this would require an infinite amount of energy for the acceleration.

NB: c is the speed of light in vacuum. In matter the speed of light v is smaller than c, characterized by the *refractive index* n as $n = c/v$. This leads e.g. to refraction by Snell's law at an interface. In matter the speed of massive particles can be larger than the speed of light there. This happens e.g. in a nuclear reactor when electrons move faster than the speed of light in water ($0.75c$). As water is a dielectric, the light waves generated from the response to the moving charge lag behind and a phenomena similar to a sonic boom is created. This phenomenon is termed Cherenkov radiation. If you have the opportunity to see it in a nuclear reactor, we highly recommend to take it. The color is a very intense deep blue.

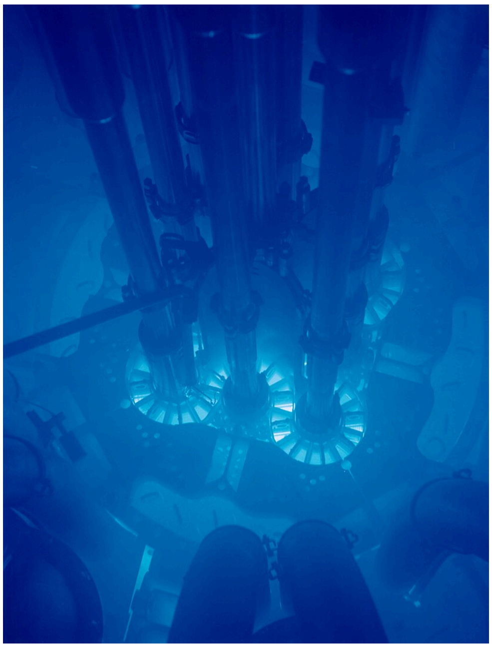


Figure 3.89: Cherenkov radiation glowing in the core of the Advanced Test Reactor at Idaho National Laboratory (Wikipedia Commons, CC BY-SA 2.0)

Exercise 3.90: 🌶️

Observer S and S' are connected via a Lorentz Transformation of the form

$$
\begin{aligned}
ct' &= \gamma\left(ct - \frac{V}{c}x\right) \\
x' &= \gamma\left(x - \frac{V}{c}ct\right)
\end{aligned}
\tag{3.139}
$$

with $V/c = 12/13$.

S' observes a particle of mass m traveling in the positive x'-direction with velocity $U'/c = 40/41$.

Find, using the 4-velocity, the velocity of m according to S.

Exercise 3.91: 🌶️

Observer S and S' are connected via a Lorentz Transformation of the form

$$
\begin{aligned}
ct' &= \gamma\left(ct - \frac{V}{c}x\right) \\
x' &= \gamma\left(x - \frac{V}{c}ct\right)
\end{aligned}
\tag{3.140}
$$

with $V/c = 12/13$.

S' observes a particle of mass m traveling in the positive y'-direction with velocity $U'/c = 40/41$.

Find, using the 4-velocity, the velocity of m according to S.

3.5.8 Exercises, examples & solutions

Updated: 04 feb 2026

Exercise 3.92: 🌶️ 🌶️

According to S' a photon is emitted at $t' = 0$ from position $L_0 = 1$ ls. It has a frequency f_0. S' is traveling at $V/C = 3/5$ in the positive x-direction with respect to S. They have synchronized their clocks when their origins coincide. Determine the time of detection of the photon by S' and by S. Find the frequency that S measures.

Exercise 3.93: 🌶️ 🌶️ 🌶️

In this exercise, the photon is emitted to S' a photon over the y'-axis. It has again a frequency f_0. S' is traveling at $V/C = 3/5$ in the positive x-direction with respect to S. They have synchronized their clocks when their origins coincide.

Find the frequency that S measures and the angle the traveling photon makes with the x-axis.

Solution 3.94: Solution to Exercise 1

According to S'

$$U'_0 = \gamma(U')c = \frac{41}{9}c$$
$$U'_1 = \gamma(U')U' = \frac{40}{9}c \tag{3.141}$$

LT to S using $\gamma(V) = \frac{13}{5}$:

$$U_0 = \gamma(V)\left(U'_0 + \frac{V}{c}U'_1\right)) = \frac{13}{5}\left(\frac{41}{9}c + \frac{12}{13}\frac{40}{9}c\right) = \frac{1013}{45}c$$
$$U_1 = \gamma(V)\left(U'_1 + \frac{V}{c}U'_0\right)) == \frac{13}{5}\left(\frac{40}{9}c + \frac{12}{13}\frac{41}{9}c\right) = \frac{1012}{45}c \tag{3.142}$$

We find u_x by taking the ratio $\frac{U_1}{U_0} = \frac{\gamma(U)u}{\gamma(U)c}$:

$$u_x = \frac{1012}{1013}c < 1$$
$$u_y = u_z = 0 \tag{3.143}$$

Solution 3.95: Solution to Exercise 2

According to S'

$$\begin{aligned}
{U'}_0 &= \gamma(U')c = \frac{41}{9}c \\
{U'}_1 &= 0 \\
{U'}_2 &= \gamma(U')U' = \frac{40}{9}c
\end{aligned} \tag{3.144}$$

LT naar S using $\gamma(V) = \frac{13}{5}$:

$$\begin{aligned}
U_0 &= \gamma(V)\left({U'}_0 + \frac{V}{c}{U'}_1\right)) = \frac{13}{5}\left(\frac{41}{9}c + 0\right) = \frac{533}{45}c \\
U_1 &= \gamma(V)\left({U'}_1 + \frac{V}{c}{U'}_0\right)) == \frac{13}{5}\left(0 + \frac{12}{13}\frac{41}{9}c\right) = \frac{492}{45}c \\
U_2 &= {U'}_2 = \frac{40}{9}c
\end{aligned} \tag{3.145}$$

We find u_x by taking the ratio $\frac{U_1}{U_0} = \frac{\gamma(U)u_x}{\gamma(U)c}$:

$$u_x = \frac{492}{533}c \tag{3.146}$$

Similarly:

$$u_y = \frac{U_2}{U_0} = \frac{\gamma(U)u_y}{\gamma(U)c} = \frac{40}{533}c \tag{3.147}$$

The magnitude of the velocity according to $S4$ is

$$u = \sqrt{u_x^2 + u_y^2} = \sqrt{\frac{243664}{284089}}c \approx 0.93c < 1c \tag{3.148}$$

Solution 3.96: Solution to Exercise 3

According to S' the photon is send at $E_1 : (c{t'}_1, {x'}_1) = (0, 1)ls$. Thus, it is received at $E_2 : (c{t'}_2, {x'}_2) = (1, 0)$. Hence, for S event E_1 has coordinates:

$$\begin{aligned}
ct_1 &= \frac{5}{4}\left(0 + \frac{3}{5}1\right) = \frac{3}{4}ls \\
x_1 &= \frac{5}{4}\left(1 + \frac{3}{5}0\right) = \frac{5}{4}ls
\end{aligned} \tag{3.149}$$

and thus, S receives this photon at $(ct_3, x_3) = (2, 0)ls$.

For S' the 4-Momentum of the photon is: $\left(\frac{hf_0}{c}, -\frac{hf_0}{c}\right)$. If we transform this to the frame of S, we get:

$$\frac{hf}{c} = \frac{5}{4}\left(\frac{hf_0}{c} + \frac{3}{5} \cdot -\frac{hf_0}{c}\right) = \frac{1}{2}\frac{hf_0}{c} \Rightarrow f = \frac{1}{2}f_0 \tag{3.150}$$

Solution 3.97: Solution to Exercise 4

In this case for S' the 4-momentum of the photon is:

$$P'^{\mu} = \left(\frac{hf_0}{c}, 0, \pm\frac{hf_0}{c}, 0\right) \tag{3.151}$$

If we translate this to the world of S, we need to realize that momentum is a vector and that the spatial parts, i.e. P^1, P^2, P^3 form a 3-vector. In this case, there is no z-component and we can write the x and y-components as the length of the vector times a cos and a sin, respectively:

$$\begin{aligned}\frac{hf}{c} &= \frac{5}{4}\left(\frac{hf_0}{c} + \frac{3}{5}0\right) = \frac{5}{4}\frac{hf_0}{c}\\ \frac{hf}{c}\cos\alpha &= \frac{5}{4}\left(0 + \frac{3}{5}\frac{hf_0}{c}\right) = \frac{3}{4}\frac{hf_0}{c}\\ \frac{hf}{c}\sin\alpha &= \pm\frac{hf_0}{c}\end{aligned} \tag{3.152}$$

Thus, from the time-like component we conclude: $f = \frac{5}{4}f_0$. This should be in agreement with the spatial components. Let's check:

$$\begin{aligned}\frac{h^2f^2}{c^2} &= \frac{h^2f^2}{c^2}\cos^2\alpha + \frac{h^2f^2}{c^2}\sin^2\alpha\\ &= \frac{3^2}{4^2}\frac{h^2f_0^2}{c^2} + \frac{h^2f_0^2}{c^2}\\ &= \frac{5^2}{4^2}\frac{h^2f_0^2}{c^2}\end{aligned} \tag{3.153}$$

Indeed, the two spatial components are in agreement with the time-like one.

Finally, we have that according to S, the photon travels at an angle $\tan\alpha = \pm\frac{4}{3} \rightarrow \alpha = \pm 53.13°$ with the x-axis.

3.5.8.1 Exercises

3.5.8.2 Answers

3.6 Relativistic dynamics and collisions

Updated: 04 feb 2026

3.6.1 4-force

In the previous chapter we have seen that 4-momentum is defined by taking the derivative of the 4-velocity with respect to proper time: $P^\mu \equiv \frac{dU^\mu}{d\tau}$. This way, 4-momentum became a 4-vector that transforms according to the Lorentz Transformation.

In Special Relativity, we deal with inertial observers. The particles they encounter can, however, accelerate under the influence of forces. As momentum is now a 4-vector, we need to define a 4-force. Following Newton, momentum changes due to a force: $\frac{d\vec{p}}{dt} = \vec{F}$. In chapter 2 we discussed Newton's second Law in the form $\vec{F} = m\vec{a}$. We saw that the acceleration did not provide any problems: we had rulers and clocks. Hence, we could measure the acceleration using know and measurable concepts like position, distance and time.

The connection between force and acceleration is of a different nature: it is a physical law, i.e. a formulation that reflects how we think nature works at its principle level. It is a hypothesis; something we need to check over and over. A rule that holds until we find inconsistencies: contradictions between theory and experiment. It takes only one experiment to overthrow a theory.

We postulate, that force is a 4-vector. Moreover, we require that in the limit of $v/c \ll 1$, we recover Newton's second Law from the spatial components of our new 4-vector force law. After all, for low velocities, Classical Mechanics of Newton and Galilei works like a charm. This indicates that we need to differentiate 4-momentum with respect to time. But, if we require force to be a 4-vector, we need to differentiate with respect to proper time. Thus, we postulate:

$$\vec{F} = \frac{d\vec{P}}{d\tau} = \gamma(u)\frac{d}{dt}(m\gamma(u)c, m\gamma(u)\vec{u}) \tag{3.154}$$

with $E = m\gamma(u)c^2$ we can rewrite this to

$$\vec{F} = \gamma(u)\left(\frac{1}{c}\frac{dE}{dt}, \frac{d}{dt}m\gamma(u)\vec{u}\right) = \gamma(u)\left(\frac{1}{c}\frac{dE}{dt}, \vec{f}\right) \tag{3.155}$$

with the 3-force $\vec{f} = \frac{d}{dt}(m\gamma(u)\vec{u})$

3.6.1.1 Work and Impulse

How about our ideas of force performing work by that force acting over a distance or providing momentum by a force working during a time interval? These ideas and concepts still apply, but they take a relativistic form. Let's see how that works.

First, the natural extension of the definition of work is now:

$$dW = F^\mu dX^\mu \tag{3.156}$$

If we repeat what we did in chapter 4, we will replace dX^μ by $U^\mu \equiv \frac{dX^\mu}{d\tau}$ and instead of F^μ we write $\frac{dP^\mu}{d\tau}$:

$$\begin{aligned} dW &= F^\mu dX^\mu \\ &= \frac{dP^\mu}{d\tau}U^\mu d\tau \\ &= m\frac{dU^\mu}{d\tau}U^\mu d\tau \\ &= mU^\mu dU^\mu \\ &= \frac{1}{2}m \quad d(U^\mu U^\mu) \end{aligned} \tag{3.157}$$

However, $U^\mu U^\mu = \gamma^2 c^2 - \gamma^2 \vec{u} \cdot \vec{u} = c^2$. That is, it is a constant (for all inertial observers the same). Thus $dU^\mu U^\mu = 0$. And we must conclude that

$$dW = F^\mu dX^\mu = 0 \tag{3.158}$$

Surprisingly, 4-force does perform zero work, always?! It is, on second thought, less surprising. Let's see how it works out in terms of 4-momentum:

$$\begin{aligned} 0 = dW &= F^\mu dX^\mu \\ &= \frac{dP^\mu}{d\tau} dX^\mu \\ &= \gamma \frac{dP^0}{dt} c dt - \gamma \frac{dP^1}{dt} u_x dt - \gamma \frac{dP^2}{dt} u_y dt - \gamma \frac{dP^3}{dt} u_z dt \\ &= \gamma \frac{dE/c}{dt} c - \gamma \vec{u} \cdot \frac{d\gamma m \vec{u}}{dt} \end{aligned} \tag{3.159}$$

Thus we can divide γ out of this equation and write $cE/c = E$:

$$0 = \frac{dE}{dt} - \vec{u} \cdot \frac{d\gamma m \vec{u}}{dt} \Rightarrow \frac{dE}{dt} = \vec{u} \cdot \frac{d\gamma m \vec{u}}{dt} \tag{3.160}$$

But this is the relativistic equivalent of

$$\mathcal{P} \equiv \frac{dE}{dt} = \vec{f} \cdot \vec{u} \tag{3.161}$$

In words: the inner product of 3-force and 3-velocity is the power $\mathcal{P}$.

3.6.2 Collisions

We will now concentrate on collisions. From our earlier discussions, for collisions we assume that we can look 'over' the collision, that is: we apply conservation of momentum and -for elastic collisions- kinetic energy. The latter implies: no non-conservative forces that dissipate mechanical energy and the potential energy prior and after the collision is the same.

We do that also for our relativistic collisions. But, we don't require that it only holds for perfectly elastic collisions. Instead, we apply it to cases where there is no possibility to turn some of the energy involved into heat. So, we focus on collisions of elementary particles that do not convert part to their energy to heat.

The 4-momentum is conserved. For $\vec{P} = \left(\frac{E}{c}, \vec{p}\right)$ we have

$$\sum_{i,before} \vec{P}_i = \sum_{j,after} \vec{P}_j \tag{3.162}$$

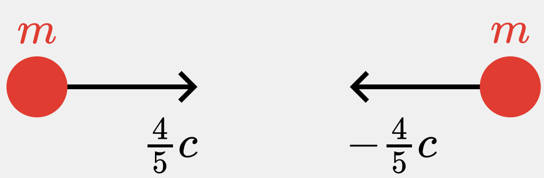
and the energy-momentum relation from the LT invariance of $\vec{P} \cdot \vec{P}$

$$E^2 = \left(mc^2\right)^2 + (pc)^2 \tag{3.163}$$

With $E = m\gamma(u)c^2$ and $\vec{p} = m\gamma(u)\vec{u}$.

Example: head on collision

Two elementary particles collide head on, see the figure below.



Both particles have mass m, after the collision there is only one particle with unknown mass M. What is the mass M and the velocity v of that one particle after the collision/fusion?

We consider the conservation of 4-momentum, in 1D:

$$\begin{aligned} P^{\mu}_{before} &= (m\gamma(u)c, m\gamma(u)u) + (m\gamma(-u)c, -m\gamma(-u)u) \\ &= (2m\gamma(u)c, 0) \\ P^{\mu}_{after} &= (M\gamma(v)c, M\gamma(v)v) \end{aligned} \tag{3.164}$$

with $\gamma(u) = \gamma(-u)$. The 4-momentum is conserved per component, from the space component we see $0 = M\gamma(v)v \Rightarrow v = 0$. With $\gamma(u) = 5/3$ and $\gamma(v) = 1$ we find for the time-component $2m\frac{5}{3} = M$.

This leads to $M = \frac{10}{3}m > 2m$. Thus, the energy prior to the collision was composed of energy associated with the masses themselves and with kinetic energy. After the collision, there is no kinetic energy but their is mass-energy and there is more of this than prior to the collision.

Example: decay of a photon into an electron and positron

We discuss if a photon (of sufficient energy $E > 1024$ keV) can decay into an electron e^- and positron e^+.

If we place us in the Center of Mass (CM) frame of the electron e^- and positron e^+ after the decay, then the total spatial momentum is $\vec{p} = 0$. The momentum before the decay of the photon is $\vec{p} = \frac{hf}{c} > 0$ therefore the decay cannot happen in free space. Momentum must be transferred to an additional different particle.

$$\left(\frac{E_e}{c}, \vec{p}\right) + \left(\frac{E_p}{c}, -\vec{p}\right) \neq \left(\frac{hf}{c}, \frac{hf}{c}\right) \tag{3.165}$$

Example: Electron-positron annihilation

We consider an electron and positron annihilation, resulting in two photons (after the collision). Remember that the decay cannot happen into one photon as shown above (Remember: equations are invariant under time reversal).

In the CM of the e^-e^+ system we have for the 4-momentum before

$$P^{\mu}_{before} = (m_e\gamma(u)c, m_e\gamma(u)u, 0, 0) + (m_e\gamma(-u)c, -m_e\gamma(-u)u, 0, 0) \tag{3.166}$$

After we have two photons, with different frequencies f, f' and traveling in different directions $\hat{s}$, $\hat{s}'$

$$P^{\mu}_{after} = \left(\frac{hf}{c}, \frac{hf}{c}\hat{s}\right) + \left(\frac{hf'}{c}, \frac{hf'}{c}\hat{s}'\right) \tag{3.167}$$

From the conservation of 4-momentum we have

$$\begin{aligned} 2m_e\gamma(u)c &= \tfrac{hf}{c} + \tfrac{hf'}{c} \\ 0 &= \tfrac{hf}{c}\hat{s} + \tfrac{hf'}{c}\hat{s}' \end{aligned} \tag{3.168}$$

From the second equation we see

$$\frac{hf}{c}\hat{s} = -\frac{hf'}{c}\hat{s}' \Rightarrow \hat{s} = -\hat{s}', \quad f = f' \tag{3.169}$$

The two photons are emitted in opposite directions (in the CM system) with the same frequency.

Filling this into the first equation $hf = m_e\gamma(u)c^2 \approx m_e c^2 = 512$ keV. The speed in the CM frame is typically $u \ll c \Rightarrow \gamma(u) = 1$.

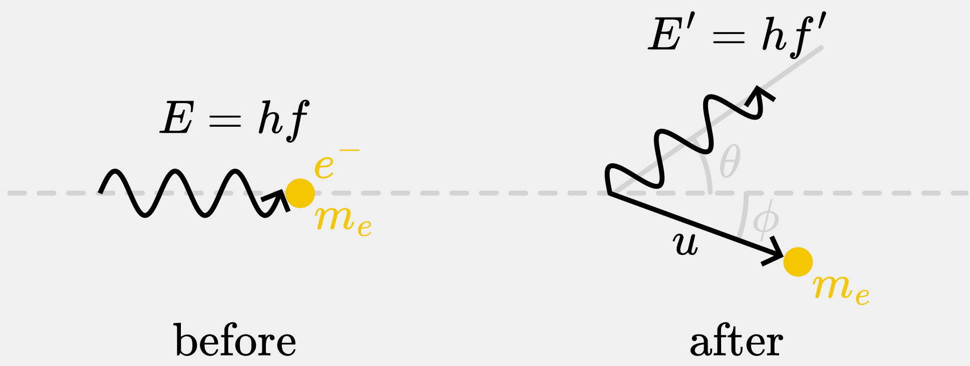
NB: please observe that analysis in the CM frame is often a good idea.

Example: Compton scattering

Compton scattering describes the (elastic) scattering of an incoming photon by a (bound) charged particle, typical an electron.

Compton scattering was discovered in 1923 by Arthur Compton. He was investigating the interaction between X-rays (that is high energy photons) and some of the light elements. The classical theory of scattering of electro-magnetic waves with matter could not explain the observations. Compton had to combine quantum mechanics with special relativity to understand the interaction of a high-energy photon and a (loosely) bound electron in the outer shell of an atom. It earned him the Nobel Prize in Physics in 1927.

When a high-frequency photon scatters at an electron, it looses some of its energy. Consequently its frequency reduces and it s wave length increases. As energy is conserved, the lost energy shows up in the electron that is emitted, that is 'freed' from the atom.



In its simplest form, we ignore the energy that is needed to free the electron from its nucleus. This is to a certain extent justified as the outer electron has a very small binding energy, that is small compared to the energy transferred between the photon and the electron.

In the rest frame of the electron, we have for the 4 different 4-momenta:

$$\begin{aligned} P_{e,b} &= (m_e c, 0, 0, 0) \\ P_{\gamma,b} &= (E/c, E/c, 0, 0) \\ P_{e,a} &= \left(\tfrac{E_{e'}}{c}, m_e\gamma(u)u\cos\phi, -m_e\gamma(u)u\sin\phi, 0\right) \\ P_{\gamma,a} &= \left(\tfrac{E\prime}{c}, \tfrac{E\prime}{c}\cos\theta, \tfrac{E\prime}{c}\sin\theta, 0\right) \end{aligned} \tag{3.170}$$

We have

$$P_{e,b} + P_{\gamma,b} = P_{e,a} + P_{\gamma,a} \tag{3.171}$$

Now we make use of the LT invariance of $\vec{P}^2$

$$\left(P_{e,b} + P_{\gamma,b} - P_{\gamma,a}\right)^2 = P_{e,a}^2 \tag{3.172}$$

$$P_{e,b}^2 + P_{\gamma,b}^2 + P_{\gamma,a}^2 + 2P_{e,b}P_{\gamma,b} - 2P_{e,b}P_{\gamma,a} - 2P_{\gamma,b}P_{\gamma,a} = P_{e,a}^2 \tag{3.173}$$

where we know $P_{e,b}^2 = P_{e,a}^2 = m_e^2c^2$ (totally elastic collision) and $P_\gamma^2 = 0$ directly as shown before. Evaluating the cross terms gives

$$m_e^2c^2 + 0 + 0 + 2m_eE' - 2m_eE - 2\frac{EE'}{c^2}(1-\cos\theta) = m_e^2c^2 \tag{3.174}$$

We isolate the energy after the collision E'

$$E' = \frac{Em_ec^2}{m_ec^2 + E(1-\cos\theta)} \tag{3.175}$$

With $E = hc/\lambda$ we obtain

$$\frac{\lambda'}{hc} = \frac{m_ec^2 + \frac{hc}{\lambda}(1-\cos\theta)}{\frac{hc}{\lambda}m_ec^2} \tag{3.176}$$

Now we only multiply both sides by hc and on the right we divide out, to obtain

$$\lambda' = \lambda + \frac{h}{m_ec}(1-\cos\theta) \tag{3.177}$$

Alternatively, we could try and solve the collision by directly using conservation of momentum. This is much more work than the P^2 trick. The calculation goes as follows.

In the rest frame of the electron

$$P^\mu_{before} = \left(\frac{hf}{c}, \frac{hf}{c}, 0, 0\right) + (m_ec, 0, 0, 0) \tag{3.178}$$

After the scattering

$$\begin{aligned} P^\mu_{after} = &\left(\frac{hf'}{c}, \frac{hf'}{c}\cos\theta, \frac{hf'}{c}\sin\theta, 0\right) + \\ &+(m_e\gamma(u)c, m_e\gamma(u)u\cos\phi, -m_e\gamma(u)u\sin\phi, 0) \end{aligned} \tag{3.179}$$

We have 3 equations, but 4 unknowns (f', u, ϕ, θ). Therefore the outgoing frequency f' is not uniquely determined, but dependent on the scattering angle θ. We can eliminate 2 (here u, ϕ) of the 4 unknowns, to remain with a relation for the other two.

For the spatial momentum we have

$$\begin{aligned} \tfrac{hf}{c} &= \tfrac{hf\prime}{c}\cos\theta + m_e\gamma(u)u\cos\phi \\ 0 &= \tfrac{hf\prime}{c}\sin\theta - m_e\gamma(u)u\sin\phi \end{aligned} \tag{3.180}$$

We rewrite the equations slightly, before squaring them and then adding them to eliminate ϕ

$$\begin{aligned} \tfrac{hf}{c} - \tfrac{hf\prime}{c}\cos\theta &= m_e\gamma(u)u\cos\phi \\ \tfrac{hf\prime}{c}\sin\theta &= m_e\gamma(u)u\sin\phi \end{aligned} \tag{3.181}$$

We indeed eliminate ϕ to

$$\frac{h^2f^2}{c^2} - 2\frac{hfhf'}{c^2}\cos\theta + \frac{h^2f'^2}{c^2} = m_e^2\gamma^2(u)u^2 \quad (*) \tag{3.182}$$

The right hand side of the equation is the space component squared of the momentum after: $p^2_{e\prime} = m_e^2\gamma^2(u)u^2$, but this can be related to the energy via the momentum-energy relation for the moment after $(p_{e\prime}c)^2 = E^2_{e\prime} - (m_ec^2)^2$. We will use this to eliminate the unknown speed u.

The energies can be related via the 0-component of the 4-momentum

$$\begin{aligned} \tfrac{hf}{c} + m_ec &= \tfrac{hf\prime}{c} + \tfrac{E\prime_e}{c} \\ \Rightarrow E_e'^2 &= (hf - hf' + m_ec^2)^2 \end{aligned} \tag{3.183}$$

Substituting the energy $E_e'^2$ into the momentum-energy relation and replacing the right hand side of equation $(*)$ after multiplying by c^2 to

$$h^2 f^2 - 2hfhf' \cos\theta + h^2 f'^2 = \left(hf - hf' + m_e c^2\right)^2 - \left(m_e c^2\right)^2 \tag{3.184}$$

Indeed we have removed the speed u and angle ϕ. We cannot do more, but remain with a relation for the frequency f' after scattering as function of angle θ. To this end we evaluate the square in the equation, cancel a few terms and rearrange to

$$\begin{aligned} 2hfm_e c^2 - 2hf'm_e c^2 &= 2h^2 ff'(1-\cos\theta) \\ \tfrac{c}{f'} - \tfrac{c}{f} &= \tfrac{h}{m_e c}(1-\cos\theta) \end{aligned} \tag{3.185}$$

Finally, by replacing the frequency with the wavelength $f\lambda = f'\lambda' = c$

$$\lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta) \tag{3.186}$$

This is, of course, the same result as we derived earlier.

As $\cos\theta < 1$ we find $\lambda' > \lambda$, which makes sense as the photon can only loose energy to the electron in the initial rest frame of the electron. After the scattering the electron can pick up some speed.

To analyze the outcome we check for

- $\theta = 0$ (no scattering): $\Rightarrow \lambda' = \lambda$ which makes sense
- $\theta = \pi$: backwards scattering, maximal $\Delta\lambda = \frac{2h}{m_e c}$ largest energy transfer

3.6.3 Exercises, examples & solutions

Updated: 04 feb 2026

3.6.3.1 Worked Examples

Example: Momentum of an accelerated electron

Momentum of an accelerated electron: compute the momentum and speed of an electron after acceleration in a potential of $V = 300\ \ \mathrm{kV}$.

From $E^2 = (mc^2)^2 + (pc)^2$ we have $p = \frac{1}{c}\sqrt{E^2 - (mc^2)^2}$ and using $E = mc^2 + E_{kin}$ we have

$$p = \frac{1}{c}\sqrt{2mc^2 E_{kin} + E_{kin}^2} \tag{3.187}$$

With $E_{kin} = 300$ keV and $m_e = 511$ keV. The speed can be computed from rearranging $E_{kin} = mc^2(\gamma - 1)$ to $\frac{v}{c} = \sqrt{1 - \frac{(mc^2)^2}{(E_{kin}+mc^2)^2}} = \sqrt{1 - \frac{511^2}{811^2}} = 0.77$. Please observe how practical it is to use the units eV!

Example: Decay of a neutral kaon

Decay of a neutral kaon into three pions. $K^0 \to \pi^- + \pi^+ + \pi^0$. Show that the three pions trajectories are in one plane.

In the rest frame of the kaon we have $\vec{p}_K = 0$ before the decay. By conservation of momentum we have after the decay $\vec{p}_{\pi^-} + \vec{p}_{\pi^+} + \vec{p}_{\pi^0} = 0$. A necessary and sufficient condition for three vectors $\vec{p}_1, \vec{p}_2, \vec{p}_3$ to lie in one plane is that $\vec{p}_1 \cdot (\vec{p}_2 \times \vec{p}_3) = 0$ (Remember that this expression gives the volume of the parallelepiped spanned by the three vectors). From the conservation of momentum we have $\vec{p}_1 = -\vec{p}_2 - \vec{p}_3$. Now we can compute $(-\vec{p}_2 - \vec{p}_3) \cdot (\vec{p}_2 \times \vec{p}_3) = -\vec{p}_2 \cdot (\vec{p}_2 \times \vec{p}_3) - \vec{p}_3 \cdot (\vec{p}_2 \times \vec{p}_3) = 0$. The two terms are each zero individually as the term in the bracket is perpendicular to $\vec{p}_2$ and $\vec{p}_3$ respectively.

If the trajectories in the rest frame of the kaon are in one plane, then they are also in one plane in all other frames. A coordinate transformation only shifts or rotates, which transfers a plane into a plane, but does not e.g. shear or bend a plane.

Exercise 3.100: 🌶️

A particle of mass M disintegrates into two fragments. In the rest frame of M, fragment m_1 has a mass of $\frac{1}{4}M$ and a velocity $u_1/c = 3/5$.

Find the mass and velocity of the other fragment.

Exercise 3.101: 🌶️

A particle of mass m is initially at rest (in frame S). A photon of frequency f is traveling over the x-axis and will be absorbed by the particle. Another photon is emitted. This photon is also traveling over the x- axis but in the opposite direction as incoming photon.

The incoming photon energy equals mc^2, the emitted photon $\frac{1}{4}mc^2$. Find the velocity and mass of the particle after the process of absorbing and emitting the photons.

Exercise 3.102: 🌶️ 🌶️

An elementary particle of mass M moves in the frame of observer S with a velocity $v/c = 12/13$. The particle is unstable and decays into a particle of mass m and a photon. The particle m has velocity $u/c = 4/5$. Both M and m move along the x-axis in the positive direction.

1. Find the mass m in terms of M.
2. What is the frequency of the photon?

Exercise 3.103: 🌶️ 🌶️

A particle of mass m moves with velocity $v_1/c = 1/2$ in the positive direction over the x-axis. At the same time, an identical particle is moving with the same velocity in the positive y-direction over the y-axis. At some point in time the two particles collide and as a result a new particle of mass M is formed.

Find the mass and velocity-vector of the new particle.

Exercise 3.104: 🌶️ 🌶️ 🌶️

A particle of mass $\frac{3}{5}m$ is moving at velocity $v_1/c = 4/5$ over the $x - axis$. From the other side a particle with mass $\frac{4}{5}m$ is approaching with velocity $v_2/c = 3/5$. The two particles will collide. After the collision, they maintained each their original mass. The collision is perfectly elastic.

1. Find the velocities of both masses in the world of Galilei and Newton.
2. The same but now in the world of Lorentz and Einstein.

Solution 3.105: Solution to Exercise 1

Prior to the disintegration, particle M has 4-momentum:

$$P_i^\mu = (Mc, 0) \tag{3.188}$$

After the disintegration, we have two particles with 4-momentum:

$$P_{1,a}^\mu = \left(\frac{1}{4}M\frac{5}{4}c, \frac{1}{4}M\frac{5}{4}\frac{3}{5}c\right) \tag{3.189}$$

and

$$P_{2,a}^\mu = (m_2\gamma_2 c, m_2\gamma_2 u_2) \tag{3.190}$$

From conservation of momentum we get:

$$\begin{aligned} 1 &= \frac{5}{16} + \frac{m_2}{M}\gamma_2 \to \frac{m_2}{M}\gamma_2 = \frac{11}{16} \\ 0 &= \frac{3}{16} + \frac{m_2}{M}\gamma_2\frac{u_2}{c} \to \frac{m_2}{M}\gamma_2\frac{u_2}{c} = -\frac{3}{16} \end{aligned} \tag{3.191}$$

Take the ratio of the last two equations:

$$\frac{u_2}{c} = -\frac{3}{11} \tag{3.192}$$

and from this we find

$$\frac{m_2}{M} = \frac{4\sqrt{7}}{16} \tag{3.193}$$

Thus, we see that the mass after the disintegration is $\frac{1}{4}M + \frac{4\sqrt{7}}{16}M \approx 0.911 < M$.

Solution 3.106: Solution to Exercise 2

Before the absorption of the photon the 4-momentum is:

$$P_i^\mu = \left(\frac{hf}{c}, \frac{hf}{c}\right) + (mc, 0) = (2mc, mc) \tag{3.194}$$

After emitting the photon, the particle has mass M and velocity u. The emitted photon has as frequency $\tilde{f}$ and 4-momentum $\left(\frac{h\tilde{f}}{c}, -\frac{h\tilde{f}}{c}\right) = \left(\frac{1}{4}mc, -\frac{1}{4}mc\right)$. The total momentum after the process is:

$$P_f^\mu = \left(\frac{1}{4}mc + M\gamma c, -\frac{1}{4}mc + M\gamma u\right) \tag{3.195}$$

Since 4-momentum is conserved, we find:

$$\begin{aligned} 2mc &= \frac{1}{4}mc + M\gamma c \\ mc &= -\frac{1}{4}mc + M\gamma u \end{aligned} \tag{3.196}$$

We rearrange the two above equations:

$$\begin{aligned} M\gamma c &= \frac{7}{4}mc \\ M\gamma u &= \frac{5}{4}mc \end{aligned} \tag{3.197}$$

If we divide the second equation by the first, we have:

$$\frac{u}{c} = \frac{5}{7} \tag{3.198}$$

The mass of the particle is:

$$M = \frac{7}{4\gamma}m = \frac{1}{2}\sqrt{6}\,m \tag{3.199}$$

Solution 3.107: Solution to Exercise 3

Initially, the 4-Momentum is

$$P_i^\mu = (M\gamma(v)c, M\gamma(v)v) \tag{3.200}$$

with

$$\frac{v}{c} = \frac{12}{13} \to \gamma(v) = \frac{13}{5} \tag{3.201}$$

After the decay, we have

$$P_f^\mu = \left(m\gamma(u)c + \frac{hf}{c}, m\gamma(u)u + \frac{hf}{c}\hat{f}\right) \tag{3.202}$$

with $\hat{f}$ a unit vector pointing in the $\pm x$-direction. We know $\frac{u}{c} = \frac{4}{5} \to \gamma(u) = \frac{5}{3}$. Conservation of 4-momentum now leads to:

$$\begin{aligned} \frac{5}{3}mc + \frac{hf}{c} &= \frac{13}{5}Mc \\ \frac{4}{3}mc \pm \frac{hf}{c} &= \frac{12}{5}Mc \end{aligned} \tag{3.203}$$

We still need to find out which direction the photon travels: parallel to m or in the opposite direction. According to the above conservation of 4-momentum both seem possible. We require that in the above $f > 0$.

First we inspect the negative sign of $\pm$:

$$\begin{aligned} \frac{5}{3}mc + \frac{hf}{c} &= \frac{13}{5}Mc \\ \frac{4}{3}mc - \frac{hf}{c} &= \frac{12}{5}Mc \end{aligned} \tag{3.204}$$

If we solve for f, we get $f < 0$, which conflicts our requirement. That leaves us with the $+$sign:

$$\begin{aligned} \frac{5}{3}mc + \frac{hf}{c} &= \frac{13}{5}Mc \\ \frac{4}{3}mc + \frac{hf}{c} &= \frac{12}{5}Mc \end{aligned} \tag{3.205}$$

Solving for m gives: $m = \frac{3}{5}M$. If we plug this back in, we find for the photon $hf = \frac{8}{5}Mc^2$.

Solution 3.108: Solution to Exercise 4

The total 4-momentum before the collision is

$$P_i^\mu = \left(2m\gamma c, \frac{1}{2}m\gamma c, \frac{1}{2}m\gamma c\right) with \gamma = \frac{2}{3}\sqrt{3} \tag{3.206}$$

After the collision, we have only one particle with 4-momentum

$$P_f^\mu = \left(M\gamma_f c, M\gamma_f u_x, M\gamma_f u_y\right) with \gamma_f = \frac{1}{\sqrt{1-\frac{u_x^2+u_y^2}{c^2}}} \tag{3.207}$$

In this process, 4-momentum is conserved.

From P^1 and P^2 we get

$$\begin{aligned} \frac{1}{2}m\gamma c &= M\gamma_f u_x \\ \frac{1}{2}m\gamma c &= M\gamma_f u_y \end{aligned} \tag{3.208}$$

hence, $u_x = u_y$. The new particle moves over the line $x = y$.

If we combine P^0 with P^1, we find:

$$\begin{aligned} 2m\gamma c &= M\gamma_f c \\ \frac{1}{2}m\gamma c &= M\gamma_f u_x \end{aligned} \tag{3.209}$$

This gives $\frac{u_x}{c} = \frac{1}{4}$. Thus, the new particle moves with velocity $\vec{u} = \frac{1}{4}c\hat{x} + \frac{1}{4}c\hat{y}$. We find its mass by calculating $\gamma_f = \frac{1}{\sqrt{1-2\frac{1}{16}}} = 2\sqrt{\frac{2}{7}}$ and using this in the P^0 equation:

$$2m\gamma c = M\gamma_f c \to M = \sqrt{\frac{14}{3}}m \tag{3.210}$$

Solution 3.109: Solution to Exercise 5

a. In classical mechanics, we use -for this type of collision- conservation of momentum and of kinetic energy. This gives us:

$$\begin{aligned} p: \quad \frac{3}{5}m\frac{4}{5}c - \frac{4}{5}m\frac{3}{5}c = \frac{3}{5}mu + \frac{4}{5}mU \rightarrow U = -\frac{3}{4}u \\ E_{kin}: \quad \frac{1}{2}\frac{3}{5}m\left(\frac{4}{5}c\right)^2 + \frac{1}{2}\frac{4}{5}m\left(\frac{3}{5}c\right)^2 = \frac{1}{2}\frac{3}{5}mu^2 + \frac{1}{2}\frac{4}{5}mU^2 \end{aligned} \tag{3.211}$$

This set has as solution (not surprising): $u = -\frac{4}{5}c, U = \frac{3}{5}c$.

b. Now we use 4-momentum conservation:

$$P_i^\mu = \left(\frac{3}{5}m\frac{5}{3}c, \frac{3}{5}m\frac{5}{3}\frac{4}{5}c\right) + \left(\frac{4}{5}m\frac{5}{4}c, -\frac{4}{5}m\frac{5}{4}\frac{3}{5}c\right) = \left(2mc, \frac{1}{5}mc\right) \tag{3.212}$$

Note: the spatial part of momentum is thus non-zero, in contrast to the classical case.

After the collision we have:

$$P_f^\mu = \left(\frac{3}{5}m\gamma_1 c, \frac{3}{5}m\gamma_1 u\right) + \left(\frac{4}{5}m\gamma_2 c, -\frac{4}{5}m\gamma_2 U\right) \tag{3.213}$$

with

$$\gamma_1 = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} and \gamma_2 = \frac{1}{\sqrt{1-\frac{U^2}{c^2}}} \tag{3.214}$$

Next, we use conservation of 4-momentum: $P_i^\mu = P_f^\mu$. This is, however, hard to do analytical! Thus we use either a graphical or numerical method. If you do this, you will find:

$$u = -0.7355c \quad and \quad U = +0.8050c \tag{3.215}$$

3.6.3.2 Exercises

3.6.3.3 Answers

4. Appendix

Updated: 04 feb 2026 some text

4.1 Relevant math

Updated: 04 feb 2026 Rather than describing in detail the required mathematical concepts, we here provide a summary of the main concepts, together with some code snippets to illustrate how to implement them in Python. We align with the approach in the TU Delft books:

- Linear Algebra
- Calculus However, here we use the notations and quantities used in this book.

4.1.1 Calculus

Updated: 04 feb 2026

4.1.1.1 Derivatives

$$f(x) = x^2 \Longrightarrow \frac{d}{dx} f(x) = 2x \tag{4.1}$$

or in general:

$$f(x) = x^n \Longrightarrow \frac{d}{dx} f(x) = n x^{n-1} \tag{4.2}$$

Three perspectives on the derivative (werk van .. in TPT):

1. **Geometric**: The derivative at a point gives the slope of the tangent line to the curve at that point.
2. **Physical**: The derivative represents the instantaneous rate of change, such as velocity being the derivative of position with respect to time.
3. **Analytical**: The derivative is defined as the limit of the average rate of change as the interval approaches zero.

From

Grasple 1

4.1.1.1.1 Chain rule

A function with a function $f(g(x))$ (like $\sin(x^2)$) can be differentiated using the chain rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) \tag{4.3}$$

Example: Chain rule example

Differentiate $h(x) = \sin(x^2)$.

Here we first identify our two functions ($f(g(x)) = \sin(g(x))$ and $g(x) = x^2$). Using the chain rule we get:

$$h'(x) = f'(g(x)) \cdot g'(x) = \cos(x^2) \cdot 2x = 2x \cos(x^2) \tag{4.4}$$

4.1.1.1.2 Product rule

A function that is the product of two functions $f(x)$ and $g(x)$ can be differentiated using the product rule:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x) \tag{4.5}$$

A simple example would be differentiating $h(x) = x \cdot x^2$ where we know this could be written as $h(x) = x^3$ and its derivative is $h'(x) = 3x^2$. Using the product rule we get:

$$h'(x) = \frac{d}{dx}[x \cdot x^2] = 1 \cdot x^2 + x \cdot 2x = x^2 + 2x^2 = 3x^2 \tag{4.6}$$

Example: Product rule example

Differentiate $h(x) = x^3 \cdot \sin(x)$.

Here we identify our two functions ($f(x) = x^3$ and $g(x) = \sin(x)$). Using the product rule we get:

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) = 3x^2 \cdot \sin(x) + x^3 \cdot \cos(x) \tag{4.7}$$

4.1.1.1.3 Quotient rule

The quotient rule is applied when differentiating a function that is the quotient of two functions $f(x)$ and $g(x)$: $h(x) = \frac{f(x)}{g(x)}$. The derivative is then give by:

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2} \tag{4.8}$$

Example: Quotient rule example

Differentiate $h(x) = \frac{x^2}{\sin(x)}$.

Here we identify our two functions ($f(x) = x^2$ and $g(x) = \sin(x)$). Using the quotient rule we get:

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2} = \frac{2x \cdot \sin(x) - x^2 \cdot \cos(x)}{\sin^2(x)} \tag{4.9}$$

4.1.1.1.4 Summarized

Function	Derivative
$f(x) \cdot g(x)$	$f'(x)g(x) + f(x)g'(x)$
$\frac{f(x)}{g(x)}$	$\frac{g(x)f'(x) - f(x)g'(x)}{g'(x)^2}$
$f(g(x))$	$f'(g(x))g'(x)$

4.1.1.1.5 List of standard derivatives

Below is a list of some standard functions $Z(x, y)$ and their derivatives with respect to x.

Function $Z(x, y)$	$\frac{dZ}{dx}$
$x + y$	1
$x \cdot y$	y
x^n	$n \cdot x^{n-1}$
e^{cx}	ce^{cx}
n^x	$n^x \ln n$
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$1 + \tan^2 x$

4.1.1.2 Partial derivatives

Above we have only discussed derivatives of functions with one variable. However, many functions depend on multiple variables, e.g. $Z(x, y)$. In such cases, we can compute the partial derivative with respect to one of the variables, treating the other variables as constants. Consider the function $Z(x, y) = x^2y + y^3$. The partial derivative of Z with respect to x is computed as follows:

$$\frac{\partial Z}{\partial x} = \frac{\partial}{\partial x}(x^2y + y^3) = 2xy + 0 = 2xy \tag{4.10}$$

and with respect to y:

$$\frac{\partial Z}{\partial y} = \frac{\partial}{\partial y}(x^2 y + y^3) = x^2 + 3y^2 \tag{4.11}$$

Noted that in the first year physics course, you will encounter this as well as you often have to deal with uncertainties in multiple quantities. To find the total uncertainty, you will need to compute the partial derivatives of the function with respect to each variable.

4.1.1.3 Integration

4.1.1.4 Line integrals

We encounter line integrals in chapter 4 on work. The amount of work done by a force field $\vec{F}$ when moving an object along a path C is given by the line integral:

$$W = \int_C \vec{F} \cdot d\vec{r} \tag{4.12}$$

We can visualize this by drawing the force field as arrows in space and the path as a curve. The line integral sums up the contributions of the force along the path, taking into account both the magnitude and direction of the force relative to the path. By looking whether the path goes with or against the force field, we can determine whether the work on the object or by the object.

VISUALIZATION

In a conservative force field, such as gravity or electrostatic forces, the work done is path-independent and only depends on the initial and final positions.

4.1.1.5 Closed loop integrals

We encounter closed loop integrals in chapter … There we see that a closed loop integral is the same as a line integral where the start and end point are the same. A closed loop integral is denoted as:

$$\oint_C \vec{F} \cdot d\vec{r} \tag{4.13}$$

If we use again the idea of a conservative force field, we can see that in such a field the closed loop integral is always zero, as the work done going from point A to point B is exactly canceled out by the work done going back from point B to point A.

4.1.1.6 Curl, divergence and gradient

For two or three dimensional vector fields, the derivative can apply to all directions. For this, the nabla operator ∇ exists. In three dimensions, the nabla operator is a vector that takes the partial derivative along each coordinate:

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \tag{4.14}$$

The nabla operator can be applied to a scalar (*gradient*) or to a vector(field) using the dot or cross product (*divergence* and *curl*). We will discuss each of these below.

Note that some of the explanation below is better understood when looking at linear algebra first.

4.1.1.6.1 Curl

The curl is used in the chapter on work and energy, specifically in the context of assessing whether a force field is conservative (if so, the path from A to B does not matter on the amount of work that is done). The curl of a vector field $\boldsymbol{F}$ is denoted as $\nabla \times \boldsymbol{F}$. In mathematical terms, it provides a measure of the rotation (or swirling strength) of the field at a given point. A zero curl indicates that the field is irrotational, which is a characteristic of conservative fields. To get a better conceptual understanding, we can inspect the two fields below. The left is clearly rotating around the center, while the right one is not as all vectors point to the same direction (0,0). Hence, we expect that if we take the curl of the left field, it will be non-zero, while for the right field it will be zero.

```
import numpy as np
import matplotlib.pyplot as plt


def F1(x, y):
    return y, -x

def F2(x, y):
    r = np.sqrt(x**2 + y**2)
    r = np.where(r == 0, 1e-10, r)  # Avoid division by zero
    return -x / r**3, -y / r**3

N = 7
xlim=(-2, 2)
ylim=(-2, 2)
x = np.linspace(xlim[0], xlim[1], N)
y = np.linspace(ylim[0], ylim[1], N)
X, Y = np.meshgrid(x, y)

Fx1, Fy1 = F1(X, Y)
Fx2, Fy2 = F2(X, Y)

fig, (ax1, ax2) = plt.subplots(2, 1, figsize=(5, 10))
ax1.quiver(X, Y, Fx1, Fy1)
ax2.quiver(X, Y, Fx2, Fy2)

ax1.set_xlabel("x")
ax1.set_ylabel("y")
ax1.set_xlim(xlim)
ax1.set_ylim(ylim)
ax2.set_xlabel("x")
ax2.set_ylabel("y")
ax2.set_xlim(xlim)
ax2.set_ylim(ylim)
plt.show()
```

Mathematically, the curl in three dimensions is defined as:

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{4.15}$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the unit vectors in the x, y, and z directions, respectively, and F_x, F_y, and F_z are the components of the vector field $\boldsymbol{F}$. Note that the outcome of taking the curl at a point returns a vector (or taking the curl of the vector field results in another vector field).

4.1.1.6.2 divergence

Divergence is used in chapter on … Divergence of a vector field $\boldsymbol{F}$ is denoted as $\nabla \cdot \boldsymbol{F}$. It quantifies the magnitude of a source or sink at a given point in the field. A positive divergence indicates a source (where field lines are diverging), while a negative divergence indicates a sink (where field lines are converging). Mathematically, the divergence in three dimensions is defined as:

$$\nabla \cdot \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{4.16}$$

where F_x, F_y, and F_z are the components of the vector field $\boldsymbol{F}$. Note that taking the divergence at a point returns a scalar value, not a vector.

4.1.1.6.3 Gradient

The gradient of function f is ∇f. It creates a vector that indicates the directions in which f increases or decreases:

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} \tag{4.17}$$

4.1.2 Linear Algebra

Updated: 04 feb 2026

4.1.2.1 Vectors

In the first chapter, the idea of a vector was introduced as a physical quantity that has both a magnitude and a direction. A vector in three-dimensional space can be represented as:

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{4.18}$$

Its magnitude (or length) is given by:

$$|\,|\,\vec{r}\,|\,| = \sqrt{x^2 + y^2 + z^2} \tag{4.19}$$

In Python, we can represent vectors using NumPy arrays and calculate their magnitude as follows:

```
import numpy as np
x, y, z = 3, 4, 5
r = np.array([x, y, z])
# r  = np.array([3, 4, 5])     equivalent to the to lines above
magnitude = np.linalg.norm(r)
print(magnitude)
```

4.1.2.1.1 Inner product

Suppose we have two vectors $\vec{F} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, visualized below.

What should be noted is that these two vectors are orthogonal: they don't share any direction. We know that the work done by a force in the direction of the displacement is zero if the force and displacement are orthogonal and can show this by taking the inner product. The inner product is also called the dot product and is denoted by a dot between the two vectors: $\vec{F} \cdot \vec{r}$. Mathematically, the inner product of these two vectors is defined as:

$$W = \vec{F} \cdot \vec{r} = F_x r_x + F_y r_y + F_z r_z \tag{4.20}$$

Its outcome is a scalar quantity. For our specific vectors, we have:

$$W = \vec{F} \cdot \vec{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot 2 + 1 \cdot 0 + 0 \cdot 0 = 0 \tag{4.21}$$

More formally, for any inner product in $\mathbb{R}^n$, the **inner product** of two vectors $\boldsymbol{v} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is defined as

$$\boldsymbol{v} \cdot \boldsymbol{w} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n. \tag{4.22}$$

Using Python, we can compute the inner product as follows:

```
import numpy as np
F1 = np.array([0, 1, 0])
r1 = np.array([2, 0, 0])
inner_product = np.dot(F1, r1)
print(inner_product)  # Output: 0
```

4.1.2.1.2 Cross product

The cross product was introduced in this book first in the context of torque. We had an arm $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and a force $\vec{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$ and defined the torque as the cross product of these two vectors:

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{4.23}$$

we can calculate the cross product as:

$$\vec{\tau} = \begin{pmatrix} r_y F_z - r_z F_y \\ r_z F_x - r_x F_z \\ r_x F_y - r_y F_x \end{pmatrix} \tag{4.24}$$

needs some attention

or, as a mnemonic (ezelsbruggetje), we can include the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$:

$$\vec{\tau} = \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{pmatrix} \tag{4.25}$$

The outcome of a cross product is a vector quantity, thus with both a magnitude and direction. Moreover, the direction of the resulting vector is orthogonal to both $\vec{r}$ and $\vec{F}$, following the right-hand rule. Note: $F \times r = -(r \times F)$, meaning the direction is reversed.

```python
import numpy as np
r = np.array([2, 0, 0])
F = np.array([0, 1, 0])
torque = np.cross(r, F)
print(torque)  # Output: [0 0 2]
```

4.1.2.1.3 Vector fields

A vector field, for instance a force field or an electric field, assigns a vector to every point in space. When visualized, each point in space has an arrow indicating both the direction and the magnitude of the vector. By changing the function F below, you can create your own vector fields.

4.1.2.2 Matrices

4.1.2.2.1 Matrix multiplication

A matrix can be used to transform a vector into another vector, do an operation on a vector. (Note! It has much more functionalities such as representing systems of equations, but we will not go into that here). For example, consider a matrix A and a vector $\vec{r}$:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{4.26}$$

We now can multiply the matrix A with the vector $\vec{r}$ to obtain a new vector $\vec{R}$:

$$R = A\vec{r} = \begin{pmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{pmatrix} \tag{4.27}$$

4.1.2.2.2 Rotation matrices

4.1.2.2.3 Two dimensional unit matrix

$$R = A\vec{r} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.28}$$

rotation matrix

We can introduce a rotation matrix A that rotates a vector $\vec{r}$ in two-dimensional space. Below this show for a matrix A swaps the x and y coordinates:

$$R = A\vec{r} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \tag{4.29}$$

90 degrees rotation

Given this concept we can introduce a matrix A that rotates a vector $\vec{r}$ by 90 degrees counterclockwise:

$$R = A\vec{r} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \tag{4.30}$$

```
import numpy as np
import matplotlib.pyplot as plt

F = np.array([1, 0])
A = np.array([[0, -1], [1, 0]])  # 90 degrees rotation matrix

R = A @ F                        # Matrix multiplication to rotate vector F

plt.figure(figsize=(6,6))
plt.quiver(0, 0, F[0], F[1], angles='xy', scale_units='xy', scale=1,
color='r', label='original vector')
plt.quiver(0, 0, R[0], R[1], angles='xy', scale_units='xy', scale=1,
color='b', label='rotated vector')
plt.xlim(-1, 2)
plt.ylim(-1, 2)
plt.legend()

plt.show()
```

45 degrees rotation

In a similar way, we can define a rotation matrix for a 45-degree clockwise rotation:

$$R = A\vec{r} = \begin{pmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2}(x+y) \\ \frac{\sqrt{2}}{2}(-x+y) \end{pmatrix} \tag{4.31}$$

In Python, we can implement this as follows:

```
import numpy as np
import matplotlib.pyplot as plt

F = np.array([1, 0])
angle = 45   # degrees
theta = np.radians(-angle)  # Convert angle to radians for clockwise rotation
A = np.array([[np.cos(theta), -np.sin(theta)], [np.sin(theta),
np.cos(theta)]])

R = A @ F  # Matrix multiplication to rotate vector F

plt.figure(figsize=(6,6))
plt.quiver(0, 0, F[0], F[1], angles='xy', scale_units='xy', scale=1,
color='r', label='original vector')
plt.quiver(0, 0, R[0], R[1], angles='xy', scale_units='xy', scale=1,
color='b', label='rotated vector')
plt.xlim(-1, 2)
plt.ylim(-1, 2)
plt.legend()
plt.show()
```

4.1.2.2.4 Three dimensional

$$R = A\vec{r} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{4.32}$$

$$R = A\vec{r} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ x \end{pmatrix} \tag{4.33}$$

4.1.3 Differential equations

Updated: 04 feb 2026 Already in chapter 1 we have seen the concept of a differential equation: an equation that relates a function to its derivatives. In mechanics, differential equations arise naturally when we express Newton's laws in terms of position, velocity, and acceleration.

4.1.3.1 First order

A first-order differential equation relates a function to its first derivative. The general form of a first-order differential equation is:

$$\frac{dx}{dt} = f(t) \tag{4.34}$$

If we know the function $f(t)$, we can solve for $x(t)$ by integrating both sides with respect to time:

$$x(t) = x(t_0) + \int_{t_0}^{t} f(\tau) d\tau \tag{4.35}$$

Example: Simple first-order differential equation

Consider a simple first-order differential equation where the rate of change of position is constant:

$$\frac{dx}{dt} = v_0 \tag{4.36}$$

where v_0 is a constant velocity. Integrating both sides gives:

$$x(t) = x(t_0) + v_0(t - t_0) \tag{4.37}$$

Example: First order equation

Consider know a function that changes with time:

$$\frac{dx}{dt} = -2t + 1 \tag{4.38}$$

To find the function $x(t)$, we can integrate both sides with respect to time:

$$x(t) = x(t_0) + \int_{t_0}^{t} (-2\tau + 1) d\tau = x(t_0) + \left[-\tau^2 + \tau\right]_{t_0}^{t} = x(t_0) + (-t^2 + t) - (-t_0^2 + t_0) \tag{4.39}$$

Note that these two examples are dependent on a function that is itself only dependent on time. More complex first-order differential equations can involve the function itself, such as:

$$\frac{dx}{dt} = -kx(t) \tag{4.40}$$

where k is a constant. This type of equation often arises in contexts such as radioactive decay or cooling processes. We can solve this equation using separation of variables or integrating factors, leading to an exponential solution $x(t) = Ae^{-kt}$. We can verify this solution by differentiating it:

$$\frac{dx}{dt} = -kAe^{-kt} = -kx(t) \tag{4.41}$$

Slotting in this equation in the above allows us to solve for the constant A based on initial conditions:

$$\frac{dx}{dt} = -kx(t) \Longrightarrow x(t) = x(t_0)e^{-k(t-t_0)} \tag{4.42}$$

4.1.3.2 Second order

In a second order differential equation, the function is related to its second derivative. The general form of a second-order differential equation is:

$$\frac{d^2x}{dt^2} = f(t, x) \tag{4.43}$$

A familiar example from mechanics is Newton's second law in relation to a mass-spring system:

$$F_{net} = ma = m\frac{d^2x}{dt^2} = -kx \tag{4.44}$$

We know this leads to simple harmonic motion, with the general solution:

$$x(t) = A\cos(\omega t) + B\sin(\omega t) \tag{4.45}$$

where $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency, and A and B are constants determined by initial conditions.

4.1.4 Series

Updated: 04 feb 2026

4.1.4.1 Taylor series

Every function can be approximated by a polynomial, called its Taylor series. The Taylor series of a function $f(x)$ around the point a is given by:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \tag{4.46}$$

We can visualize this idea by looking at the Taylor series expansion of the sine function. The Taylor series of a sine function is given by:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \tag{4.47}$$

or more generally:

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \tag{4.48}$$

Below we can see what happens when we take more and more terms of the Taylor series expansion of the sine function. The black dots represent the actual sine function, while the colored lines represent the Taylor series approximations with increasing numbers of terms.

```
import numpy as np
import matplotlib.pyplot as plt
import ipywidgets as widgets
from ipywidgets import interact
import math

x = np.linspace(0,2*np.pi,1000)
y = np.sin(x)

def taylor_series(x, n):
    ts = np.zeros_like(x)
    for k in range(int(n)):
        ts += ((-1)**k * x**(2*k+1)) / math.factorial(2*k+1)
    return ts

def update(n):
    plt.clf()
    plt.figure()

    plt.plot(x,y,'k.')
    plt.plot(x,taylor_series(x,n))

    plt.ylim(-5,5)
    plt.show()

# Use FloatSlider for smooth interaction
interact(update, n=widgets.FloatSlider(min=1, max=20, step=1, value=4))
```

4.1.4.2 Fourier series

4.2 Example Exam

Updated: 04 feb 2026

4.2.1 Question 1

A point particle with mass $2m$ and velocity $6v$ moves along the x-axis (in the positive x-direction). Another point particle with mass 3m and velocity v also moves along the x-axis (also in the positive direction). Both particles will collide at some point. This is a 1-dimensional problem.

1. (2 points) Give the definition of the velocity and position of the center of mass of a two-particle system. Determine the velocity of the center of mass of the two-particle system from this problem.
2. (3 points) Give the Galilean Transformation and apply it so that the problem can be analyzed in the center of mass frame. Give the velocities of both particles in the center of mass frame.
3. (3 points) Give the velocities of the particles after the collision as seen from the center of mass frame, given that it is a perfectly elastic collision.
4. (2 points) Transform your solution back to the original frame and give the velocities after the collision.

4.2.2 Question 2

Given a 1D force $F(x) = -F_0 \sin\left(\frac{2\pi x}{L}\right)$.

1. (2 points) Determine the corresponding potential $V(x)$ and sketch the potential.
2. (2 points) Determine the equilibrium points of $F(x)$ and determine whether these are stable or unstable points.

A point mass m is located at position $x = 0$. At a certain time, the point mass receives a small displacement $\Delta x > 0$ (with $\Delta x \ll L$).

1. (2 points) Set up the equation of motion for point mass m and approximate it for the small displacement Δx.
2. (2 points) Show that point mass m will perform harmonic oscillation. Also determine the oscillation frequency.
3. (2 points) If point mass m had received an initial velocity v_0 instead of an initial displacement: how large must v_0 be at minimum for m to escape to infinity? Explain your answer.

4.2.3 Question 3

Two point masses (each mass m) are connected by a massless rope of constant length L via a pulley at a horizontal distance κ. Mass 1 lies on a flat, horizontal surface and can move frictionless over it. Mass 2 hangs freely from the rope. The weight acts vertically downward on mass 2. The rope experiences no friction. See figure.

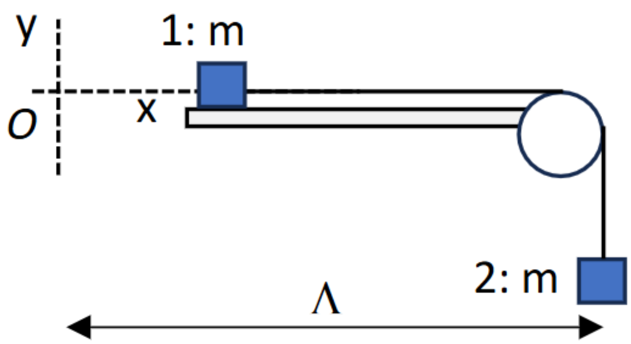


Figure 4.2: The x and y axes are represented as dotted lines. The origin of the coordinate system is O.

At time $t = 0$ both masses have velocity 0. The rope is taut at all times.

1. (2 points) Draw a free body diagram of this system showing the forces acting on each of the masses.
2. (2 points) Give the definition of angular momentum and give N_2 for the angular momentum.
3. (2 points) Set up the total angular momentum of this system with respect to the given origin O.
4. (2 points) Set up N2 for the angular momentum of this system.
5. (2 points) Calculate the acceleration of both masses.

4.2.4 Question 4

Observer S’ moves relative to observer S with speed $V/c = 3/5$ in the positive x-direction. The x and x’ axes are parallel. When the origins of S and S’ coincide, the clocks in S and S’ are set to �� = ��’ = 0.

At c�� = 1���� a photon with frequency f_0 is emitted from the origin of S that moves in the positive x-direction. This is event E1.

The photon is detected by S’ (at her origin) at some later time. This is event E2.

The photon reflects subsequently on a mirror that is at rest in S’. For S’ this means that the reflected photon has the same frequency as the photon before the reflection.

1. (1pt) Give the Lorentz Transformation and calculate the gamma factor for this problem.
2. (3pt) Determine the coordinates of E1 according to S’. Also determine the coordinates of E2 for both S and S’.
3. (2pt) Calculate the length of the interval E2-E1 and show that S and S’ find the same value. Classify the interval.
4. (1pt) Determine the frequency of the photon at E2 according to S’.
5. (1pt) Determine the frequency of the reflected photon according to S.

4.2.5 Qestion 5

Observer S’ moves relative to observer S with speed $V/c = 3/5$ in the positive x-direction. The x and x’ axes are parallel. When the origins of S and S’ coincide, the clocks in S and S’ are set to �� = ��’ = 0.

In the frame of S’, a mass m with speed $4/5c$ collides (relativistically) with a stationary particle with mass $5m$. After the collision there is 1 particle. a. (1pt) Give the definition of four-momentum for a particle with mass m and speed u. b. (2pt) Determine the speed and mass of the particle after the collision according to S’. c. (2pt) Determine the speed of the particles before the collision according to S.

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