# Classical Mechanics & Special Relativity for Starters


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# Contents

# 1. Introduction

Updated: 18 jan 2026 This book provides an introduction for freshman students into the world of classical mechanics and special relativity theory. Much of physics is build on the basic ideas from classical mechanics. Hence an early introduction to the topic can be beneficial for new students. However, at the start of studying physics, lots of the required math is not available yet. That means that all kind of concepts that are very useful cannot be invoked in the lectures and teaching. That does not have to be a disadvantage. It can also be used to help the students by introducing some math and coupling it directly to the physics, making more clear why mathematics should be studied and what its ‘practical use’ is. With this book, we aim to introduce new students directly at the start of their studies into the world of physics, more specifically the world of Newton, Galilei and many others who laid the foundation of physics. We aim to help students getting a good understanding of the theory, i.e. the framework of physics. What is ‘work’ and why do we use it? Why is kinetic energy $\frac{1}{2}mv^2$ and not $\frac{1}{3}mv^2$ or $\frac{1}{2}mv^3$? Both 3′s are fundamentally wrong, but what is behind it?

## 1.1 About this book

Classical mechanics is the starting point of physics. Over the centuries, via Newton’s three fundamental laws formulated around 1687, we have built a solid framework describing the material world around us. On these pages, you will find a textbook with animations, demos, interactive elements and exercises for studying introductory classical mechanics. Moreover, we will consider the first steps of Einstein’s Special Theory of Relativity published 1905.

This material is made to support first year students from the BSc Applied Physics at Delft University of Technology during their course *Classical Mechanics and Relativity Theory*, MechaRela for short. But, of course, anybody interested in Classical Mechanics and Special Relativity is more than welcome to use this book.

With this e-book our aim is to provide learning material that is:

- self-contained
- easy to modify and thus improve over the years
- interactive by providing demos, interactive elements and exercises next to the lectures

This book is based on Mudde & Rieger 2025.

That book was already beyond introductory level and presumed a solid basis in both calculus and basic mechanics. All texts in this book were revised, additional examples and exercises were included, picture and drawings have been updated and interactive materials have been included. Hence, this book should be considered a stand-alone new version. Note that we made good use of other open educational resources, several exercises stem from such resources. Where we use external materials, we acknowledge and credit the original sources.

### 1.1.1 Special features

In this book you will find some ‘special’ features. Some of these are indicated by their own formatting:

Exercise 1.1: 🌶️

Each chapter includes a variety of exercises tailored to the material. We distinguish between exercises embedded within the instructional text and those presented on separate pages. The in-text exercises should be completed while reading, as they offer immediate feedback on whether the concepts and mathematics are understood. The separate exercise sets are intended for practice after reading the text and attending the lectures.

To indicate the level of difficulty, each exercise is marked with 1, 2, or 3 🌶️

**Intermezzos**

Intermezzos contain background information on the topic or on the people that worked on relevant concepts.

**Experiments**

We include some basic experiments that can be done at home.

**Example: Examples**

We provide various examples showcasing, e.g., calculations.

**Python**

We include in-browser python code, as well as downloadable .py files which can be executed locally. If there is an in-browser, press the ON-button to 'enable compute'. Try it by pushing the ON-button and subsequently the play button and see the output in de code-cell below.

```
print("The square root of 2 is: ", 2 ** 0.5)
```

The interactive elements, such as Python code, hover over functionality, embedded youtube clips etc, only work in the online environment, not in the pdf or printed book. Where possible we included qr codes and links to the online clips. We also include interactive exercises made with Grasple. These exercises provide immediate feedback on your answers, allowing you to learn from your mistakes and deepen your understanding of the material. Here is an example exercise:

New concepts, such as *Free body diagram*, are introduced with a hover-over. If you move your mouse over the italicized part of the text, you will get a short explanation.

You have the opportunity to download some of the materials as Jupyter Notebook file and play with the code offline. We advise you to use Jupyter Lab in combination with MyST.

### 1.1.2 Feedback

This is the first version (second cycle) of this book. Although many have worked on it and several iterations have been made, there might still be issues. Do you see a mistake? Do you have suggestions for exercises? Are parts missing or abundant? Tell us! You can use the `Feedback` button in the top right of the screen. You will need a (free) GitHub account to report an issue!

## 1.2 Authors

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Special thanks to Hanna den Hertog for (re)making most of the drawings, Luuk Fröling for his technical support and Dion Hoeksema for converting the .js scripts to .py files. Also thanks to Vebe Helmes, Alexander Lopes-Cardozo, Sep Schouwenaar, Alesja Zorina, Winston de Greef and Boas Bakker for their comments and suggestions.

## 1.3 Open Educational Resource



This website is a Jupyter Book. Source files are available for download using the button on the top right.

### 1.3.1 Software and license

This website is a Jupyter Book. Markdown source files are available for download using the button on the top right, licensed under CC-BY-NC (unless stated otherwise). All python codes / apps are freely reusable, adaptable and redistributable (CC0).

### 1.3.2 Images, videos, apps, intermezzos

The cover image is inspired by the work of 3Blue1Brown developer Grant Sanderson.

All vector images have been made by Hanna den Hertog, and are available in vector format through the repository. For reuse, adapting and redistribution, adhere to the CC-BY licences.

We embedded several clips from 3Blue1Brown in accord with their licences requirements.

The embedded vpython apps are made freely available from trinket.

Some videos from NASA are included, where we adhere to their regulations.

At various places we use pictures which are in the public domain. We comply to the regulations with regard to references.

The Intermezzos, which elaborate on the lives of various scientists and the efforts behind key physical discoveries, are composed by drawing from a range of different sources. Rather than directly reproducing any one account, these stories have been reworked into a narrative that fits the context and audience of this book.

### 1.3.3 How to cite this book

R.F. Mudde, B. Rieger, C.F.J. Pols, *Classical Mechanics & Special Relativity for Beginners*, CC BY-NC

```
@book{MuddeRiegerPols2025,
  author    = {Robert F. Mudde and Bernd Rieger and Freek Pols},
  title     = {Classical Mechanics \& Special Relativity for Beginners},
  year      = {2025},
  publisher = {TU Delft Open},
  note      = {CC BY-NC},
  url       = {https://interactivetextbooks.tudelft.nl/mecharela}
}
```

# 2. Mechanics

Updated: 18 jan 2026 In this part we cover the fundamentals of Classical Mechanics. We discuss the three laws of Newton and their first consequences. This part focusses on the primary concepts and quantities: *Force, Work, Energy, Angular Momentum.* We derive and discuss the conservation equations of these and their applications. Two topics receive special attention: *Oscillations* and *Collisions.* We restrict the discussion to one-dimensional cases as much as possible to help understand the physics and not get lost in multi-dimensional problems at an early stage. However, more-dimensionality is not avoided as, for instance, it should be clear from the start that physics not only deals with numbers (better: scalars) but equally important, if not more important, with vectors. Moreover, *angular momentum* and *torque* by their very nature require multi-dimensional thinking.

There are also subjects that we don't touch upon. We will not deal with rigid bodies (although some of the ideas are met when talking about kinetic energy: its translational versus rotational flavors). Rigid bodies require a higher level of abstract thinking and will take up quite some time that is not available in most introductory courses on Classical Mechanics. Nor will we discuss non-inertial frames of reference and fictitious forces like the centrifugal and Coriolis Force. This is left for later years. Finally, the concepts of the Lagrangian and Hamiltonian are left for an advanced course in Classical Mechanics.

## 2.1 The language of Physics

Updated: 18 jan 2026 Physics is the science that seeks to understand the fundamental workings of the universe: from the motion of everyday objects to the structure of atoms and galaxies. To do this, physicists have developed a precise and powerful language: one that combines mathematics, colloquial and technical language, and visual representations. This language allows us not only to describe how the physical world behaves, but also to predict how it will behave under new conditions.

In this chapter, we introduce the foundational elements of this language, covering how to express physical ideas using equations, graphs, diagrams, and words. You'll also get a first taste of how physics uses numerical simulations as an essential complement to analytical problem solving.

This language is more than just a set of tools—it is how physicists *think*. Mastering it is the first step in becoming fluent in physics.

### 2.1.1 Representations

Physics problems and concepts can be represented in multiple ways, each offering a different perspective and set of insights. The ability to translate between these representations is one of the most important skills you will develop as a physics student. In this section, we examine three key forms of representation: equations, graphs and drawings, and verbal descriptions using the context of a base jumper, see Figure 1.

Figure 2.1: A base jumper is used as context to get familiar with representation, picture from https://commons.wikimedia.org/wiki/File:04SHANG4963.jpg

#### 2.1.1.1 Verbal descriptions

Words are indispensable in physics. Language is used to describe a phenomenon, explain concepts, pose problems and interpret results. A good verbal description makes clear:

- What is happening in a physical scenario;
- What assumptions are being made (e.g., frictionless surface, constant mass);
- What is known and what needs to be found.

**Example: Base jumper: Verbal description**

Let us consider a base jumper jumping from a 300 m high building. We take that the jumper drops from that height with zero initial velocity. We will assume that the stunt is performed safely and in compliance with all regulations/laws. Finally, we will assume that the problem is 1-dimensional: the jumper drops vertically down and experiences only gravity, buoyancy and air-friction.

We know (probably from experience) that the jumper will accelerate. Picking up speed increases the drag force acting on the jumper, slowing the *acceleration* (meaning it still accelerates!). The speed keeps increasing until the jumper reaches its terminal velocity, that is the velocity at which the drag (+ buoyancy) exactly balance gravity and the sum of forces on the jumper is zero. The jumper no longer accelerates.

Can we find out what the terminal velocity of this jumper will be and how long it takes to reach that velocity?

#### 2.1.1.2 Visual representations

*Visual representations* help us interpret physical behavior at a glance. Graphs, motion diagrams, free-body diagrams, and vector sketches are all ways to make abstract ideas more tangible.

- **Drawings** help illustrate the situation: what objects are involved, how they are moving, and what forces act on them.
- **Graphs** (e.g., position vs. time, velocity vs. time) reveal trends and allow for estimation of slopes and areas, which have physical meanings like velocity and displacement.

**Example: Base jumper: Free body diagram**

The situation of the base jumper is sketched in Figure 2 using a Free body diagram. Note that all details of the jumper are ignored in the sketch.

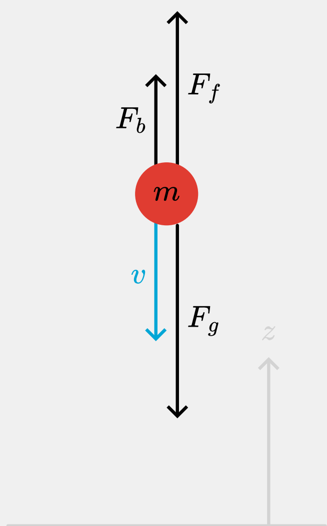



Figure 2.2: Force acting on the jumper.

- $m$ = mass of jumper (in kg);
- $v$ = velocity of jumper (in m/s);
- $F_g$ = gravitational force (in N);
- $F_f$ = drag force by the air (in N);
- $F_b$ = buoyancy (in N): like in water also in air there is an upward force, equal to the weight of the displaced air.

#### 2.1.1.3 Equations

Equations are the compact, symbolic expressions of physical relationships. They tell us how quantities like velocity, acceleration, force, and energy are connected.

**Example: Base jumper: equations**

The forces acting on the jumper are already shown in Figure 2. Balancing of forces tells us that the jumper might reach a velocity such that the drag force and buoyancy exactly balance gravity and the jumper no longer accelerates:

$$F_g = F_f + F_b \tag{2.1}$$

We can specify each of the forces:

$$\begin{aligned} F_g &= -mg = -\rho_p V_p g \\ F_f &= \frac{1}{2}\rho_{air} C_D A v^2 \\ F_b &= \rho_{air} V_p g \end{aligned} \tag{2.2}$$

with $g$ the acceleration of gravity, $\rho_p$ the density of the jumper ($\approx 10^3\ \mathrm{kg/m^3}$), $V_p$ the volume of the jumper, $\rho_{air}$ the density of air ($\approx 1.2\ \mathrm{kg/m^3}$), $C_D$ the so-called drag coefficient, $A$ the frontal area of the jumper as seen by the air flowing past the jumper.

A physicist is able to switch between these representations, carefully considering which representations suits best for the given situation. We will practice these when solving problems.

**Danger**

Note that in the example above we neglected directions. In our equations we should have been using vector notation, which we will cover in one of the next sections in this chapter.

### 2.1.2 How to solve a physics problem?

One of the most common mistakes made by ‘novices’ when studying problems in physics is trying to jump as quickly as possible to the solution of a given problem or exercise by picking an equation and slotting in the numbers. For simple questions, this may work. But when stuff gets more complicated, it is almost a certain route to frustration.

There is, however, a structured way of problem solving, that is used by virtually all scientists and engineers. Later this will be second nature to you, and you will apply this way of working automatically. It is called IDEA, an acronym that stands for:

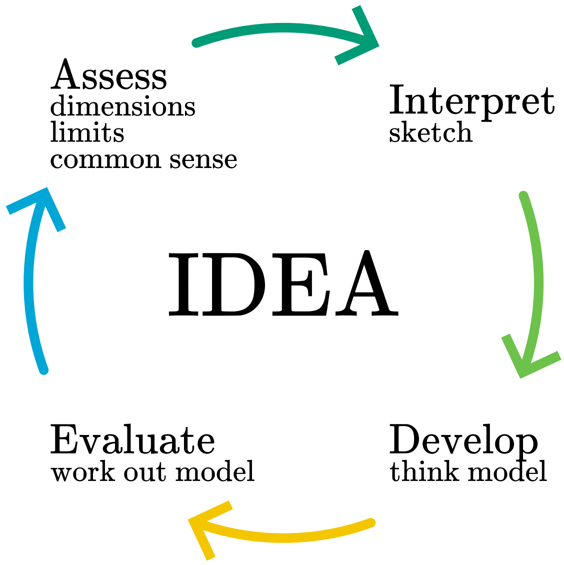



Figure 2.3: IDEA

- **Interpret** - First think about the problem. What does it mean? Usually, making a sketch helps. ***Actually:*** *always start with a sketch*;
- **Develop** - Build a model, from coarse to fine, that is, first think in the governing phenomena and then subsequently put in more details. Work towards writing down the equation of motion and boundary conditions;
- **Evaluate** - Solve your model, i.e. the equation of motion;
- **Assess** - Check whether your answer makes any sense (e.g. units OK? What order of magnitude did we expect?). Is our answer in the order of magnitude that we expected?

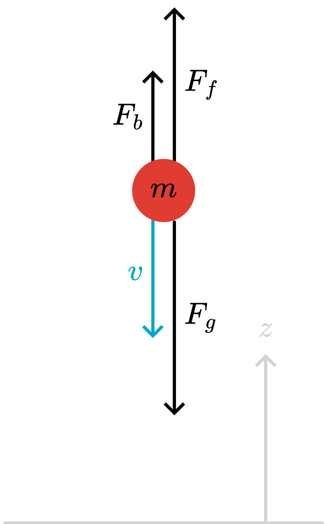
We will practice this and we will see that it actually is a very relaxed way of working and thinking. We strongly recommend to apply this strategy for your homework and exams (even though it seems strange in the beginning).

The first two steps (Interpret and Develop) typically take up most of the time spend on a problem.

#### 2.1.2.1 Example

**Interpret**

Three forces act on the jumper, shown in the figure below. Finding the terminal velocity implies that all forces are balanced ($\sum F = 0$).




The buoyancy force is much smaller than the force of gravity (about $0.1\%$) and we neglect it.

**Develop**

We know all forces: gravitational force equals the drag force

$$\begin{aligned} F_g &= F_f \\ mg &= \frac{1}{2}\rho_{air} C_D A v^2 \end{aligned} \tag{2.3}$$

**Evaluate**

Assume a mass of 75 kg, an acceleration due to gravity of 9.81 $\mathrm{m/s^2}$, and air density of 1.2 $\mathrm{kg/m^3}$, a drag coefficient of 1, a frontal surface area of 0.7 $\mathrm{m^2}$.

$$mg = \frac{1}{2}\rho_{air} C_D A v^2 \tag{2.4}$$

Rewriting:

$$\begin{aligned} v &= \sqrt{\frac{2mg}{\rho_{air} C_D A}} \\ v &= \sqrt{\frac{2 \cdot 75\ (\mathrm{kg}) \cdot 9.81\ (\mathrm{m/s^2})}{1.2\ (\mathrm{kg/m^3}) \cdot 1 \cdot 0.7\ (\mathrm{m^2})}} \\ v &= 40\ \mathrm{m/s} \end{aligned} \tag{2.5}$$

**Assess**

We may know that raindrops typically reach a terminal velocity of less than 10 m/s. A terminal velocity of 40 m/s seems therefore plausible for a much heavier object.

***Note*** that we didn't solve the problem entirely! We only calculated the terminal velocity, where the question was how long it would roughly take to reach such a velocity.

**Good Practice**

It is a good habit to make your mathematical steps small: one-by-one. Don't make big jumps or do multiple steps at once. If you make a mistake, it will be very hard to trace it back.

Next: check always the dimensional correctness of your equations: that is easy to do and you will find the majorities of your mistakes.

Finally, use letters to denote quantities, including $\pi$. The reason for this is:

- letters have meaning and you can easily put dimensions to them;
- letters are more compact;
- your expressions usually become easier to read and characteristic features of the problem at hand can be recognized.

**Powers of ten**

In physics, powers of ten are used to express very large or very small quantities compactly and clearly, from the size of atoms ($\approx 10^{-10}$ m) to the distance between stars ($\approx 10^{16}$ m). This notation helps compare scales, estimate orders of magnitude, and maintain clarity in calculations involving extreme values.

We use prefixes to denote these powers of ten in front of the standard units, e.g. km for 1000 meters, ms for milliseconds, GB for gigabyte that is 1 billionbytes. Here is a list of prefixes.

| Prefix | Symbol | Math | Prefix | Symbol | Math |
|---|---|---|---|---|---|
| Yocto | y | $10^{-24}$ | Base | • | $10^{0}$ |
| Zepto | z | $10^{-21}$ | Deca | da | $10^{1}$ |
| Atto | a | $10^{-18}$ | Hecto | h | $10^{2}$ |
| Femto | f | $10^{-15}$ | Kilo | k | $10^{3}$ |
| Pico | p | $10^{-12}$ | Mega | M | $10^{6}$ |
| Nano | n | $10^{-9}$ | Giga | G | $10^{9}$ |
| Micro | µ | $10^{-6}$ | Tera | T | $10^{12}$ |
| Milli | m | $10^{-3}$ | Peta | P | $10^{15}$ |
| Centi | c | $10^{-2}$ | Exa | E | $10^{18}$ |
| Deci | d | $10^{-1}$ | Zetta | Z | $10^{21}$ |
| Base | • | $10^{0}$ | Yotta | Y | $10^{24}$ |

**On quantities and units**

Each quantity has a unit. As there are only so many letters in the alphabet (even when including the Greek alphabet), letters are used for multiple quantities. How can we distinguish then meters from mass, both denoted with the letter 'm'? Quantities are expressed in italics ($m$) and units are not (m).

We make extensively use of the International System of Units (SI) to ensure consistency and precision in measurements across all scientific disciplines. The seven base SI units are:

| Unit | Symbol | Quantity |
|---|---|---|
| meter | m | length |
| kilogram | kg | mass |
| second | s | time |
| ampere | A | electric current |
| kelvin | K | temperature |
| mole | mol | amount of substance |
| candela | cd | luminous intensity |

All other quantities can be derived from these using dimension analysis:

$$\begin{aligned} W &= F \cdot s = ma \cdot s = m\frac{\Delta v}{\Delta t} \cdot s \\ &= [\mathrm{N}] \cdot [\mathrm{m}] = [\mathrm{kg}] \cdot [\mathrm{m/s^2}] \cdot [\mathrm{m}] = [\mathrm{kg}] \cdot \frac{[\mathrm{m/s}]}{[\mathrm{s}]} \cdot [\mathrm{m}] = \left[\frac{\mathrm{kgm^2}}{\mathrm{s^2}}\right] \end{aligned} \tag{2.6}$$

***Note:*** Newton is the person, fully written the unit N is newton, without capitalization of the first letter.

**Tip**

For a more elaborate description of quantities, units and dimension analysis, see the manual of the first year physics lab course.

### 2.1.3 Calculus

Most of the undergraduate theory in physics is presented in the language of Calculus. We do a lot of differentiating and integrating, and for good reasons. The basic concepts and laws of physics can be cast in mathematical expressions, providing us the rigor and precision that is needed in our field. Moreover, once we have solved a certain problem using calculus, we obtain a very rich solution, usually in terms of functions. We can quickly recognize and classify the core features that help us understand the problem and its solution much deeper.

Given the example of the base jumper, we would like to know the jumper's position as a function of time. We can answer this question by applying Newton's second law (though it is covered in secondary school, the next chapter explains in detail Newton's laws of motion):

$$\sum F = F_g - F_f = ma = m\frac{dv}{dt} \tag{2.7}$$

$$m\frac{dv}{dt} = mg - \frac{1}{2}\rho_{air}C_D A v^2 \tag{2.8}$$

Clearly this is some kind of differential equation: the change in velocity depends on the velocity itself ($\frac{dv}{dt} = ..v(t)$). Before we even try to solve this problem ($v(t) = ...$), we have to dig deeper in the precise notation, otherwise we will get lost in directions and sign conventions.

#### 2.1.3.1 Differentiation

Many physical phenomena are described by differential equations. That may be because a system evolves in time, or because it changes from location to location. In our treatment of the principles of classical mechanics, we will use differentiation with respect to time a lot. The reason is obviously found in Newton's $2^{nd}$ law: $F = ma$.

The acceleration $a$ is the derivative of the velocity with respect to time ($a = \frac{dv}{dt}$); velocity in itself is the derivative of position with respect to time ($v = \frac{dx}{dt}$). Or when we use the equivalent formulation with momentum: $\frac{dp}{dt} = F$. So, the change of momentum in time is due to forces. Again, we use differentiation, but now of momentum.

There are three common ways to denote differentiation. The first one is by 'spelling it out':

$$v = \frac{dx}{dt} \text{ and } a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{2.9}$$

- Advantage: it is crystal clear what we are doing.
- Disadvantage: it is a rather lengthy way of writing.

Newton introduced a different flavor: he used a dot above the quantity to indicate differentiation with respect to time. So,

$$v = \dot{x}, \text{ or } a = \dot{v} = \ddot{x} \tag{2.10}$$

- Advantage: compact notation, keeping equations compact.
- Disadvantage: a dot is easily overlooked or disappears in the writing.

Finally, in math often the prime is used: $\frac{df}{dx} = f'(x)$ or $\frac{d^2f}{dx^2} = f''(x)$. Similar advantage and disadvantage as with the dot notation.

**Important**

$$v = \frac{dx}{dt} = \dot{x} = x' \tag{2.11}$$

$$a = \frac{dv}{dt} = \dot{v} = \frac{d^2x}{dt^2} = \ddot{x} \tag{2.12}$$

It is just a matter of notation.

### 2.1.4 Definition of velocity, acceleration and momentum

In mechanics, we deal with forces on particles. We try to describe what happens to the particles, that is, we are interested in the position of the particles, their velocity and acceleration. We need a formal definition, to make sure that we all know what we are talking about.

#### 1-dimensional case

In one dimensional problems, we only have one coordinate to take into account to describe the position of the particle. Let's call that $x$. In general, $x$ will change with time as particles can move. Thus, we write $x(t)$ showing that the position, in principle, is a function of time $t$. How fast a particle changes its position is, of course, given by its velocity. This quantity describes how far an object has traveled in a given time interval: $v = \frac{\Delta x}{\Delta t}$. However, this definition gives actually the average velocity in the time interval $\Delta t$. The (momentary) velocity is defined as:

**Definition Velocity**

$$v \equiv \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{(t + \Delta t) - t} = \frac{dx}{dt} \tag{2.13}$$

Note that we here use $\equiv$ rather than $=$ to indicate that this is a definition.

Similarly, we define the acceleration as the change of the velocity over a time interval $\Delta t$: $a = \frac{\Delta v}{\Delta t}$. Again, this is actually the average acceleration and we need the momentary one:

**Definition Acceleration**

$$a \equiv \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{(t + \Delta t) - t} = \frac{dv}{dt} \tag{2.14}$$

Consequently,

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \tag{2.15}$$

Now that we have a formal definition of velocity, we can also define momentum: momentum is mass times velocity, in math:

**Definition Momentum**

$$p \equiv mv = m\frac{dx}{dt} \tag{2.16}$$

In the above, we have not worried about how we measure position or time. The latter is straight forward: we use a clock to account for the time intervals. To find the position, we need a ruler and a starting point from where we measure the position. This is a complicated way of saying the we need a coordinate system with an origin. But once we have chosen one, we can measure positions and using a clock measure changes with time.


Figure 2.5: Calculating velocity requires both position and time, both easily measured e.g. using a stopmotion where one determines the position of the car per frame given a constant time interval.

#### 2.1.4.1 Vectors - more dimensional case

Position, velocity, momentum, force: they are all *vectors*. In physics we will use vectors a lot. It is important to use a proper notation to indicate that you are working with a vector. That can be done in various ways, all of which you will probably use at some point in time. We will use the position of a particle located at point P as an example.

**Tip**

See the linear algebra book on vectors.

**Position vector**

We write the position **vector** of the particle as $\vec{r}$. This vector is a 'thing', it exists in space independent of the coordinate system we use. All we need is an origin that defines the starting point of the vector and the point P, where the vector ends.

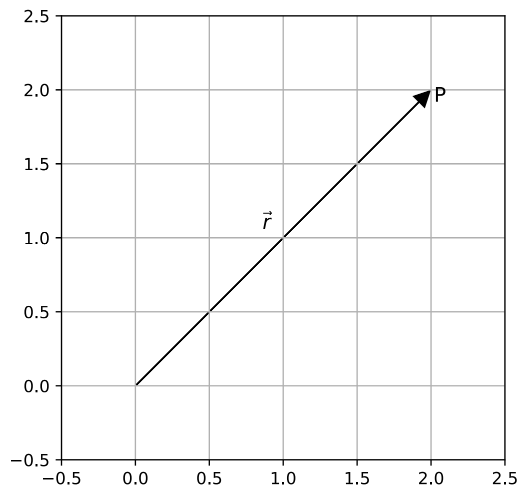



Figure 2.6: Some physical quantities (velocity, force etc) can be represented as a vector. The have in common the direction, magnitude and point of application.

A coordinate system allows us to write a representation of the vector in terms of its coordinates. For instance, we could use the familiar Cartesian Coordinate system {x,y,x} and represent $\vec{r}$ as a column.

$$\vec{r} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.17}$$

Alternatively, we could use unit vectors in the x, y and z-direction. These vectors have unit length and point in the x, y or z-direction, respectively. They are denoted in varies ways, depending on taste. Here are 3 examples:

$$\begin{aligned} &\hat{x},\ \hat{\imath},\ \vec{e}_x \\ &\hat{y},\ \hat{\jmath},\ \vec{e}_y \\ &\hat{z},\ \hat{k},\ \vec{e}_z \end{aligned} \tag{2.18}$$

With this notation, we can write the position vector $\vec{r}$ as follows:

$$\begin{aligned} \vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} \\ \vec{r} &= x\hat{\imath} + y\hat{\jmath} + z\hat{k} \\ \vec{r} &= x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \end{aligned} \tag{2.19}$$

Note that these representations are equivalent: the difference is in how the unit vectors are named. Also note, that these three representations are all given in terms of vectors. That is important to realize: in contrast to the column notation, now all is written at a single line. But keep in mind: $\hat{x}$ and $\hat{y}$ are perpendicular **vectors**.

**Other textbooks**

Note that other textbooks may use bold symbols to represent vectors:

$$\vec{F} = m\vec{a} \tag{2.20}$$

is the same as

$$\boldsymbol{F} = m\boldsymbol{a} \tag{2.21}$$

### Differentiating a vector

We often have to differentiate physical quantities: velocity is the derivative of position with respect to time; acceleration is the derivative of velocity with respect to time. But you will also come across differentiation with respect to position ($\frac{d}{dx}$).
As an example, let's take velocity. Like in the 1-dimensional case, we can ask ourselves: how does the position of an object change over time? That leads us naturally to the definition of velocity: a change of position divided by a time interval:

**Definition Velocity (Vector)**

$$\vec{v} \equiv \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt} \tag{2.22}$$

What does it mean? Differentiating is looking at the change of your 'function' when you go from $x$ to $x + dx$:

$$\frac{df}{dx} \equiv \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{2.23}$$

In 3 dimensions we will have that we go from point P, represented by $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ to 'the next point' $\vec{r} + d\vec{r}$. The small vector $d\vec{r}$ is a small step forward in all three directions, that is a bit $dx$ in the x-direction, a bit $dy$ in the y-direction and a bit $dz$ in the z-direction. Consequently, we can write $\vec{r} + d\vec{r}$ as

$$\begin{aligned} \vec{r} + d\vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} + dx\hat{x} + dy\hat{y} + dz\hat{z} \\ &= (x + dx)\hat{x} + (y + dy)\hat{y} + (z + dz)\hat{z} \end{aligned} \tag{2.24}$$

Now, we can take a look at each component of the position and define the velocity component as, e.g., in the x-direction

$$v_x = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt} \tag{2.25}$$

Applying this to the 3-dimensional vector, we get

$$\begin{aligned} \vec{v} \equiv \frac{d\vec{r}}{dt} &= \frac{d}{dt}(x\hat{x} + y\hat{y} + z\hat{z}) \\ &= \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} \\ &= v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \end{aligned} \tag{2.26}$$

Note that in the above, we have used that according to the product rule:

$$\frac{d}{dt}(x\hat{x}) = \frac{dx}{dt}\hat{x} + x\frac{d\hat{x}}{dt} = \frac{dx}{dt}\hat{x} \tag{2.27}$$

since $\frac{d\hat{x}}{dt} = 0$ (the unit vectors in a Cartesian system are constant). This may sound trivial: how could they change; they have always length 1 and they always point in the same direction. Trivial as this may be, we will come across unit vectors that are not constant as their direction may change. But we will worry about those examples later.

Now that the velocity of an object is defined, we can introduce its momentum:

**Definition Momentum (Vector)**

$$\vec{p} \equiv m\vec{v} = m\frac{d\vec{r}}{dt} \tag{2.28}$$

Albeit we have now a formal definition of momentum, we come later to its physical interpretation.

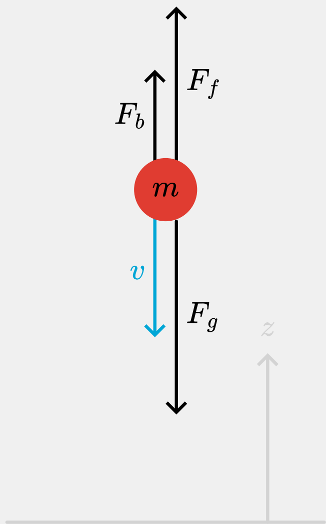
We can use the same reasoning and notation for acceleration:

**Definition Acceleration (Vector)**

$$\vec{a} \equiv \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \tag{2.29}$$

**Example: The base jumper**

Given the above explanation, we can now reconsider our description of the base jumper.




We see a z-coordinate pointing upward, where the velocity. As gravitational force is in the direction of the ground, we can state

$$\vec{F}_g = -mg\hat{z} \tag{2.30}$$

Buoyancy is clearly along the z-direction, hence

$$\vec{F}_b = \rho_{air} V g \hat{z} \tag{2.31}$$

The drag force is a little more complicated as the direction of the drag force is always against the direction of the velocity $-\vec{v}$. However, in the formula for drag we have $v^2$. To solve this, we can write

$$\vec{F}_f = -\frac{1}{2}\rho_{air} C_D A \mid v \mid \vec{v} \tag{2.32}$$

Note that $\hat{z}$ is missing in (32) on purpose. That would be a simplification that is valid in the given situation, but not in general.

### 2.1.5 Numerical computation and simulation

In simple cases we can obtain a physical model where we can derive an analytical solution. In the case of the base jumper, an analytical solution exists, though it is not trivial and requires some advanced operations as separation of variables and partial fractions:

$$v(t) = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}} t\right) \tag{2.33}$$

with

$$k = \frac{1}{2}\rho_{air} C_D A \tag{2.34}$$

In this case there is nothing to add or gain from a numerical analysis. Nevertheless, it is instructive to see how we could have dealt with this problem using numerical techniques. One way of solving the problem is, to write a computer code (e.g. in python) that computes from time instant to time instant the force on the jumper, and from that updates the velocity and subsequently the position.

```
some initial conditions
t = 0
x = x_0
v = 0
dt = 0.1

for i is 1 to N:
    compute F: formula
    compute new v: v[i+1] = v[i] - F[i]/m*dt
    compute new x: x[i+1] = x[i] + v[i]*dt
    compute new t: t[i+1] = t[i] + dt
```

You might already have some experience with numerical simulations. Figure 8 presents a script for the software Coach, which you might have encountered in secondary school.

```
'Stop condition is set
'Computations are based on Euler

x := x + flow_1*Δt1
v := v + flow_2*Δt1

t1 := t1 + Δt1

flow_1 := v
Fz := m*g
Fw := 6*d*d*v*v
f := Fz - Fw
a := f/m
flow_2 := a
```

```
t1 := 0             's
Δt1 := 0.01         's

x := 0              'm
v := 0              'm/s
m := 75             'kg
g := 9.81           'm/s^2
d := 2.5            'm

flow_1 := v         'm/s
Fz := m*g           'N
Fw := 6*d*d*v*v     'N
f := Fz - Fw        'N
a := f/m            'm/s^2
flow_2 := a         'm/s^2
```

Figure 2.8: An example of a numerical simulation made in Coach. At the left the iterative calculation process, at the right the initial conditions.

**Example: The base jumper**

Let us go back to the context of the base jumper and write some code.

First we take: $k = \frac{1}{2}\rho_{air}C_D A$ which eases writing. Newton's second law then becomes:

$$m\vec{a} = -m\vec{g} - k \mid v \mid \vec{v} \tag{2.35}$$

We rewrite this to a proper differential equation for $v$ into a finite difference equation. That is, we go back to how we came to the differential equation:

$$m \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \vec{F}_{net} \tag{2.36}$$

with $\vec{F}_{net} = -m\vec{g} - k \mid v \mid \vec{v}$

On a computer, we cannot literally take the limit of $\Delta t$ to zero, but we can make $\Delta t$ very small. If we do that, we can rewrite the difference equation (thus not taken the limit):

$$\vec{v}(t+\Delta t) = \vec{v}(t) + \frac{\vec{F}}{m}\Delta t \tag{2.37}$$

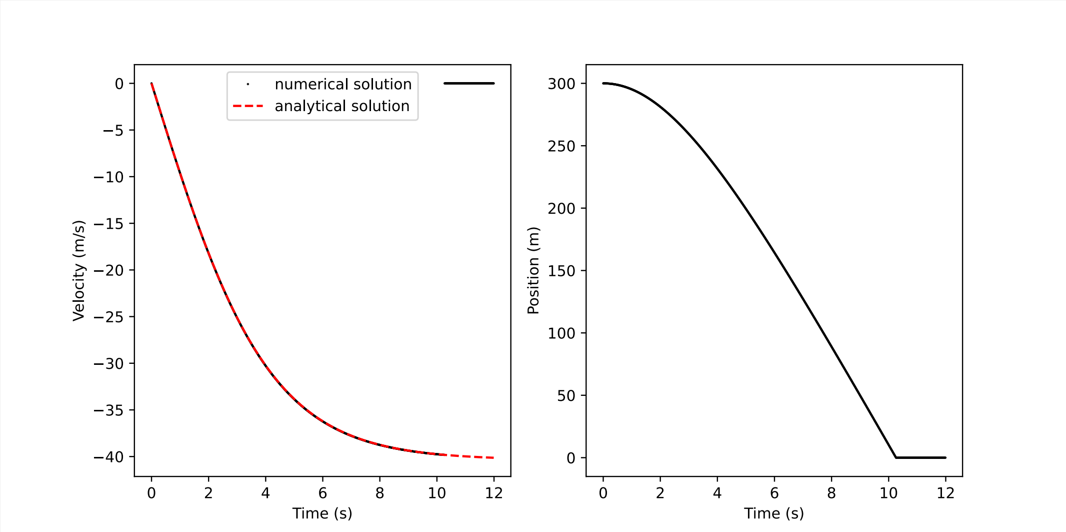
This expression forms the heart of our numerical approach. We will compute $v$ at discrete moments in time: $t_i = 0, \Delta t, 2\Delta t, 3\Delta t, \ldots$. We will call these values $v_i$. Note that the force can be calculated at time $t_i$ once we have $v_i$.

$$F_i = mg - k \mid v_i \mid v_i$$
$$v_{i+1} = v_i + \frac{F_i}{m} \Delta t \tag{2.38}$$

Similarly, we can keep track of the position:

$$\frac{dx}{dt} = v \Rightarrow x_{i+1} = x_i + v_i \Delta t \tag{2.39}$$

With the above rules, we can write an iterative code:




Important to note is the sign-convention which we adhere to. Rather than using $v^2$ we make use of $\mid v \mid v$ which takes into account that drag is always against the direction of movement. Note as well the similarity between the analytical solution and the numerical solution.

To come back to our initial problem:
It roughly takes 10 s to get close to terminal velocity (note that without friction the velocity would be 98 m/s). The building is not high enough to reach this velocity (safely).

### Exercise 2.10: Base jumper with initial velocity 🌶️

Change the code so that the base jumper starts with an initial velocity along the z-direction.

Is the acceleration in the z-direction with and without initial velocity the same? Elaborate.

Exercise 2.11: Unit analysis 🌶️

Given the formula $F = kv^2$. Derive the unit of $k$, expressed only in SI-units .

Exercise 2.12: Units based on physical constants[1] 🌶️ 🌶️

In physics, we assume that quantities like the speed of light ($c$) and Newton's gravitational constant ($G$) have the same value throughout the universe, and are therefore known as physical constants. A third such constant from quantum mechanics is Planck's constant ($\hbar$ , $h$ an with a bar). In high-energy physics, people deal with processes that occur at very small length scales, so our regular SI-units like meters and seconds are not very useful. Instead, we can combine the fundamental physical constants into different basis values.

1. Combine $c$, $G$ and $\hbar$ into a quantity that has the dimensions of length.
2. Calculate the numerical value of this length in SI units (this is known as the Planck length).
3. Similarly, combine $c$, $G$ and $\hbar$ into a quantity that has the dimensions of energy (indeed, known as the Planck energy) and calculate its numerical value.

### 2.1.6 Examples, exercises and solutions

Updated: 20 okt 2025

#### 2.1.6.1 Exercises

```
### Your code
```

```
### Your code
```

[1]Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

### Exercise 2.13: Reynolds numbers[2] 🌶️ 🌶️

Physicists often use *dimensionless quantities* to compare the magnitude of two physical quantities. Such numbers have two major advantages over quantities with numbers. First, as dimensionless quantities carry no units, it does not matter which unit system you use, you'll always get the same value. Second, by comparing quantities, the concepts 'big' and 'small' are well-defined, unlike for quantities with a dimension (for example, a distance may be small on human scales, but very big for a bacterium). Perhaps the best known example of a dimensionless quantity is the *Reynolds number* in fluid mechanics, which compares the relative magnitude of inertial and drag forces acting on a moving object:

$$\mathrm{Re} = \frac{\text{inertialforces}}{\text{dragforces}} = \frac{\rho v L}{\mu} \tag{2.40}$$

where $\rho$ is the density of the fluid (either a liquid or a gas), $v$ the speed of the object, $L$ its size, and $\mu$ the viscosity of the fluid. Typical values of the viscosity are $1.0\ \mathrm{mPa} \cdot \mathrm{s}$ for water, $50\ \mathrm{mPa} \cdot \mathrm{s}$ for ketchup, and $1.8 \cdot 10^{-5}\ \mathrm{Pa} \cdot \mathrm{s}$ for air.

1. Estimate the typical Reynolds number for a duck when flying and when swimming (you may assume that the swimming happens entirely submerged). NB: This will require you looking up or making educated guesses about some properties of these birds in motion. In either case, is the inertial or the drag force dominant?
2. Estimate the typical Reynolds number for a swimming bacterium. Again indicate which force is dominant.
3. Oil tankers that want to make port in Rotterdam already put their engines in reverse halfway across the North sea. Explain why they have to do so.
4. Express the Reynolds number for the flow of water through a (circular) pipe as a function of the diameter $D$ of the pipe, the volumetric flow rate (i.e., volume per second that flows through the pipe) $Q$, and the kinematic viscosity $\nu \equiv \eta/\rho$.
5. For low Reynolds number, fluids will typically exhibit so-called laminar flow, in which the fluid particles all follow paths that nicely align (this is the transparent flow of water from a tap at low flux). For higher Reynolds number, the flow becomes turbulent, with many eddies and vortices (the white-looking flow of water from the tap you observe when increasing the flow rate). The maximum Reynolds number for which the flow in a cylindrical pipe is typically laminar is experimentally measured to be about 2300. Estimate the flow velocity and volumetric flow rate of water from a tap with a $1.0\ \mathrm{cm}$ diameter in the case that the flow is just laminar.

### Exercise 2.14: Powers of ten 🌶️

Calculate:

1. $10^{-4} \cdot 10^{-8} =$
2. $\frac{10^{6}}{10^{-19} \cdot 10^{4}} =$
3. $10^{12} \cdot 10^{-15} =$

[2] Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

### Exercise 2.15: Moving a box 🌶

A box is on a frictionless incline of 10°. It is pushed upward with a force $F_i$ for $\Delta t = 0.5$ s. It is then moving upward (inertia) but slows down due to gravity.

Below is a part of the python code. However, some essential elements of the code are missing indicated by **(..)**.

1. Include the correct code and run it.
2. Explain the two graphs, highlighting all essential features of the graph by relating these to the given problem.
3. At what time is the acceleration 0? At what time is the box back at its origin?

The above context is not very realistic as friction is neglected. We, however, can include friction easily as it is given by $\overrightarrow{F_w} = \mu \overrightarrow{F_N}$, with $\mu = 0.05$. Note that the direction of friction changes when the direction of the velocity changes!

4. Extend the code so that friction is included.

### Exercise 2.16: Basejumper with parachute 🌶 🌶

Our base jumper has yet not a soft landing. Luckily she has a working parachute. The parachute opens in 3.8 s reaching a total frontal area of 42.6 m$^2$. We can model the drag force using $\vec{F}_{drag} = k \mid v \mid \vec{v}$ with $k = 0.37$.

Write the code that simulates this jump of the base jumper with deploying the parachute. Show the $(F_{drag}, t)$-diagram and the $(v, t)$-diagram. What is the minimal height at which the parachute should be deployed?

### Exercise 2.17: Circular motion 🌶 🌶

Remember from secondary school circular motion, where the required force is given by $F = \frac{mv^2}{r}$. The corresponding vector form is: $\vec{F} = -\frac{mv^2}{r}\hat{r}$, or equivalent: $\vec{F} = -\frac{mv^2}{r^2}\vec{r}$. Now let's simulate that motion.

Assume:

- $m = 1$ kg
- $\vec{r}_0 = (3, 0)$ m
- $\vec{v}(0) = (0, 7)$ m/s

Write the code. You know the output already (a circle with radius of 3)!

### Solution 2.18: Solution to Exercise 1

$$\begin{aligned} F &= kv^2 \\ &= [.]\left[\frac{\mathrm{m}^2}{\mathrm{s}^2}\right] \Rightarrow [.] = \left[\frac{\mathrm{kg}}{\mathrm{m}}\right] \end{aligned} \tag{2.41}$$

Solution 2.19: Solution to Exercise 2

The physical constants $c$, $G$ and $\hbar$ have the following numerical values and SI-units:

$$\begin{aligned} c &= 2.99792458 \cdot 10^8 \text{ m/s} \\ G &= 6.674 \cdot 10^-11 \text{ m}^3/(\text{kg} \cdot \text{s}^2) \\ \hbar &= 1.054 \cdot 10^{-34} \text{ kgm}^2/\text{s} \end{aligned} \tag{2.42}$$

Note: the value of $c$ is precise, i.e. by definition given this value. The second is defined via the frequency of radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom.

If we want to combine these three units into a length scale, $\mathcal{L}$, we try the following:

$$[\mathcal{L}] = [c]^A[G]^B[\quad \hbar]^C \tag{2.43}$$

What we mean here, is that the units of the quantities (denoted by [.]) left and right should be the same. Thus, we get:

$$m^1 = \left(\frac{m}{s}\right)^A \left(\frac{m^3}{kgs^2}\right)^B \left(\frac{kgm^2}{s}\right)^C \tag{2.44}$$

We try to find $A, B, C$ such that the above equation is valid. We can write this equation as:

$$m^1 = m^{A+3B+2C} \cdot kg^{-B+C} \cdot s^{-A-2B-C} \tag{2.45}$$

If we split this into requirements for m, kg, s we get:

$$\begin{aligned} m &: 1 = A + 3B + 2C \\ kg &: 0 = C - B \\ s &: 0 = -A - 2B - C \end{aligned} \tag{2.46}$$

From the second equation we get $B = C$. Substitute this into the first and third and we find:

$$\begin{aligned} m &: 1 = A + 5B \\ s &: 0 = -A - 3B \end{aligned} \tag{2.47}$$

Add these two equations: $1 = 2B \rightarrow B = \frac{1}{2}$ and thus $C = \frac{1}{2}$ and $A = -\frac{3}{2}$.

So if we plug these values into our starting equation we see:

$$\mathcal{L} = \sqrt{\frac{\hbar G}{c^3}} = 1.62 \cdot 10^{-35} \text{ m} \tag{2.48}$$

We can repeat this for energy, $\mathcal{E}$:

$$[\mathcal{E}] = [c]^\alpha[G]^\beta[\quad \hbar]^\gamma \tag{2.49}$$

Note: the unit of energy, [J] needs to be written in terms of the basic units: $[J] = \text{kgm}^2/\text{s}^2$.

The outcome is: $\alpha = \frac{5}{2}, \beta = -\frac{1}{2}, \gamma = \frac{1}{2}$ and thus our energy is:

$$\mathcal{E} = \sqrt{\frac{\hbar c^5}{G}} = 1.96 \cdot 10^9 \text{ J} \tag{2.50}$$

Solution 2.20: Solution to Exercise 3

1. The size of a duck is on the order of 30 cm. It flies at a speed of about 70 km/h, that is 20 m/s. Thus we compute for the Reynolds number of a flying duck:

$$Re \equiv \frac{\rho v L}{\mu} = 4.0 \cdot 10^5 \tag{2.51}$$

Clearly, the inertial force is dominant.

What about a swimming duck? Now the velocity is much smaller: $v \approx 1$ m/s = 3.6 km/h. The viscosity of water is $\mu_w = 1.0$ mPa $\cdot$ s and the water density is $1.0 \cdot 10^3$ kg/m$^3$. We, again, calculate the Reynolds number:

$$Re_w \equiv \frac{\rho v L}{\mu} = 3.0 \cdot 10^5 \tag{2.52}$$

Hence, also in this case inertial forces are dominant. This perhaps comes as a surprise, after all the velocity is much smaller and the viscosity much larger. However, the water density is also much larger!

2. For a swimming bacterium the numbers change. The size is now about $1\mu$m and the velocity $60\mu$ m/s (numbers taken from internet). That gives:

$$Re_b \equiv \frac{\rho v L}{\mu} = 6.0 \cdot 10^{-5} \tag{2.53}$$

and we see that here viscous forces are dominating.

3. For an oil tanker the Reynolds number is easily on the order of $10^8$. Obviously, viscous forces don't do much. An oil tanker that wants to slow down cannot do so by just stopping the motors and let the drag force decelerate them: the Reynolds number shows that the viscous drag is negligible compared to the inertial forces. Thus, the tanker has to use ots engines to slow down. Again the inertia of the system is so large, that it will take a long time to slow down. And a long time, means a long trajectory.
4. For the flow of water through a (circular) pipe the Reynolds number uses as length scale the pipe diameter. We can relate the velocity of the water in the pipe tot the total volume that is flowing per second through a cress section of the pipe:

$$Q = \frac{\pi}{4} D^2 v \to v = \frac{4Q}{\pi D^2} \tag{2.54}$$

Thus we can also write $Re$ as:

$$Re \equiv \frac{\rho v D}{\mu} = \frac{4Q}{\pi \frac{mu}{\rho} D^2} = \frac{4Q}{\pi \nu D^2} \tag{2.55}$$

1. If $Re = 2300$ for the pipe flow, we have:

$$Re = \frac{vD}{\nu} = 2300 \to v = \frac{2300\nu}{D} \tag{2.56}$$

with $\nu = 1.0 \cdot 10^{-6}$ m$^2$/s and $D = 1.0 \cdot 10^{-2}$ m we find: $v = 0.23$ m/s and $Q = 1.8 \cdot 10^{-5}$ m$^3$/s = 0.018 liter/s.

**Solution 2.21: Solution to Exercise 4**

1. $= 10^{-12}$
2. $= 10^{21}$
3. $= 10^{-3}$

Solution 2.22: Solution to Exercise 5

```
# Moving a box

## Importing libraries
import numpy as np
import matplotlib.pyplot as plt


part_4 = 1 # Turn to 0 for first part

## Constants
m = 2 #kg
F = 30 #N
g = 9.81 #m/s^2
theta = np.deg2rad(10) #degrees

mu = 0.02
F_N = m*g*np.cos(theta) #N

## Time step
dt = 0.01 #s
t = np.arange(0, 10, dt) #s
t_F_stop = 0.5

## Initial conditions
x = np.zeros(len(t)) #m
v = np.zeros(len(t)) #m/s


## Loop to calculate position and velocity
for i in range(0, len(t)-1):
    if t[i] < t_F_stop:
        a = F/m - g*np.sin(theta) - F_N*mu*np.where(v[i] != 0,
np.sign(v[i]), 0)*part_4
    else:
        a = -g*np.sin(theta) - F_N*mu*np.where(v[i] != 0, np.sign(v[i]),
0)*part_4
    v[i+1] = v[i] + a*dt
    x[i+1] = x[i] + v[i]*dt

## Plotting results
figs, axs = plt.subplots(1, 2, figsize=(10, 5))

axs[0].set_xlabel('Time (s)')
axs[0].set_ylabel('Velocity (m/s)')
axs[0].plot(t, v, 'k.', markersize=1)


axs[1].set_xlabel('Time (s)')
axs[1].set_ylabel('Position (m)')
axs[1].plot(t, x, 'k.', markersize=1)



plt.show()
```

Solution 2.23: Solution to Exercise 6

```python
# Simulation of a base jumper

## Importing libraries
import numpy as np
import matplotlib.pyplot as plt

## Constants
A = 0.7 #m^2
m = 75 #kg
k = 0.37 #kg/m
g = 9.81 #m/s^2

## Time step
dt = 0.01 #s
t = np.arange(0, 12, dt) #s

## Initial conditions
z = np.zeros(len(t)) #m
v = np.zeros(len(t)) #m/s
z[0] = 300 #m

## Deploy parachute
A_max = 42.6 #m^2
t_deploy_start = 2 #s
dt_deploy = 3.8 #s

## Loop to calculate position and velocity
for i in range(0, len(t)-1):
    F = - m*g - k*A*abs(v[i])*v[i]  #N
    v[i+1] = v[i] + F/m*dt #m/s
    z[i+1] = z[i] + v[i]*dt #m
    # Check if the jumper is on the ground
    if z[i+1] < 0:
        break
    # Deploy parachute
    if t[i] > t_deploy_start and t[i] < t_deploy_start + dt_deploy:
        A += (A_max - A)/dt_deploy*dt

## Plotting results
figs, axs = plt.subplots(1, 2, figsize=(10, 5))

axs[0].set_xlabel('Time (s)')
axs[0].set_ylabel('Velocity (m/s)')

axs[0].plot(t, v, 'k.', markersize=1, label='numerical solution')
axs[0].vlines(t_deploy_start, v[t==t_deploy_start],0, color='gray',
linestyle='--', label='parachute deploy')

axs[0].legend()

axs[1].set_xlabel('Time (s)')
axs[1].set_ylabel('Position (m)')

axs[1].plot(t, z, 'k.', markersize=1)
axs[1].vlines(t_deploy_start, 150,300, color='gray', linestyle='--',
label='parachute deploy')


plt.show()
```

Solution 2.24: Solution to Exercise 7

```
import numpy as np
import matplotlib.pyplot as plt

F = 49/3
m1 = 1
dt = 0.001
t = np.arange(0, 100, dt) # s

x1 = np.zeros(len(t)) # m
x1[0] = 3
y1 = np.zeros(len(t)) # m
vx = 0
vy = 7


for i in range(0, len(t)-1):
    ax = -F*(x1[i]-0)/np.sqrt(x1[i]**2 + y1[i]**2)/m1
    ay = -F*(y1[i]-0)/np.sqrt(x1[i]**2 + y1[i]**2)/m1
    vx = vx + ax*dt
    vy = vy + ay*dt
    x1[i+1] = x1[i] + vx*dt
    y1[i+1] = y1[i] + vy*dt


plt.figure(figsize=(4,4))
plt.plot(x1, y1, 'k.', markersize=1)
plt.xlabel('x (m)')
plt.ylabel('y (m)')
plt.show()
```

### 2.1.6.2 Solutions

## 2.2 Newton's Laws

Updated: 18 jan 2026 Now we turn to one of the most profound breakthroughs in the history of science: the laws of motion formulated by Isaac Newton. These laws provide a systematic framework for understanding how and why objects move. They form the backbone of classical mechanics. Using these three laws we can predict the motion of a falling apple, a car accelerating down the road, or a satellite orbiting Earth (though some adjustments are required in this context to make use of e.g. GPS!). More than just equations, they express deep principles about the nature of force, mass, and interaction.

In this chapter, you will begin to develop the core physicist's skill: building a simplified model of the real world, applying physical principles, and using mathematical tools to reach meaningful conclusions.

### 2.2.1 Newton's Three Laws

Much of physics, in particular Classical Mechanics, rests on three laws that carry Newton's name:

**Newton's first law (N1)**

*Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare.*

If no force acts on an object, the object moves with constant velocity.

**Newton's second law (N2)**

*Mutationem motus proportionalem esse vi motrici impressæ, & fieri secundum lineam rectam qua vis illa imprimitur.*

If a (net) force acts on an object, the momentum of the object will change according to:

$$\frac{d\vec{p}}{dt} = \vec{F} \tag{2.57}$$

**Newton's third law (N3)**

*Actioni contrariam semper & æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales & in partes contrarias dirigi.*

If object 1 exerts a force $\vec{F}_{12}$ on object 2, then object 2 exerts a force $\vec{F}_{21}$ equal in magnitude and opposite in direction on object 1:

$$\vec{F}_{21} = -\vec{F}_{12} \tag{2.58}$$

N1 has, in fact, been formulated by Galileo Galilei. Newton has, in his N2, build upon it: N1 is included in N2, after all:
if $\vec{F} = 0$, then $\frac{d\vec{p}}{dt} = 0 \to \vec{p} = constant \to \vec{v} = constant$, provided $m$ is a constant.

Most people know N2 as

$$\vec{F} = m\vec{a} \tag{2.59}$$

For particles of constant mass, the two are equivalent:
if $m = constant$, then

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a} \tag{2.60}$$

Nevertheless, in many cases using the momentum representation is beneficial. The reason is that momentum is one of the key quantities in physics. This is due to the underlying conservation law that we will derive in a minute. Momentum is a more fundamental concept in physics than acceleration. That is another reason why physicists prefer the second way of looking at forces.

Moreover, using momentum allows for a new interpretation of force: force is that quantity that - provided it is allowed to act for some time interval on an object - changes the momentum of that object. This can be formally written as:

$$d\vec{p} = \vec{F}dt \leftrightarrow \Delta\vec{p} = \int \vec{F}dt \tag{2.61}$$

The latter quantity $\vec{I} \equiv \int \vec{F}dt$ is called the impulse.

**Note**

**Momentum** is in Dutch **impuls**; the English **impulse** is in Dutch **stoot**.

In Newton's laws, velocity, acceleration and momentum are key quantities. We repeat here their formal definition.

**Definition**

$$\begin{aligned} \text{velocity} : \vec{v} &\equiv \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt} \\ \text{acceleration} : \vec{a} &\equiv \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} \\ \text{momentum} : \vec{p} &\equiv m\vec{v} = m\frac{d\vec{r}}{dt} \end{aligned} \tag{2.62}$$

**Exercise 1:** 🌶️

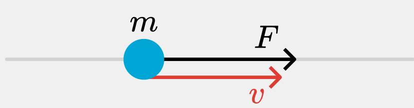
Consider a point particle of mass m, moving at a velocity $v_0$ along the x-axis. At $t = 0$ a constant force acts on the particle in the positive x-direction. The force lasts for a small time interval $\Delta t$.

What is the velocity of the particle for $t > \Delta t$ ?

**Solution to Exercise 1:** 🌶️

**Interpret**

First we make a sketch.




This is obviously a 1-dimensional problem. So, we can leave out the vector character of e.g. the force.

**Develop**

We will use $dp = Fdt$:

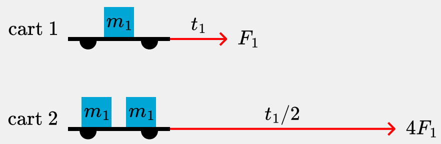
$$dp = Fdt \Rightarrow \Delta p = \int_0^{\Delta t} Fdt = F\Delta t \rightarrow \tag{2.63}$$

$$p(\Delta t) = p(0) + F\Delta t = mv_0 + F\Delta t \rightarrow \tag{2.64}$$

$$v(\Delta t) = v_0 + \frac{F}{m}\Delta t \tag{2.65}$$

Note that this example could also be solved by N2 in the form of $F = ma$. It is merely a matter of taste.

**Exercise 2: A pushing contest**




**Exercise 3: Newton's third law**

The base jumper from chapter 1 just jumped from the tall building. According to Newton's third law there are two coupled forces. Which are these, and what is the consequence of these two forces?

**Solution to Exercise 3: Newton's third law**

The gravitational force acts from the earth on the jumper. Newton's law states that the jumper thus acts a gravitational force on the earth. Hence, the earth accelerates towards the jumper!

Although this sounds silly, when comparing this idea to the sun and the planets, we must draw the conclusion that the sun is actually wobbling as it is pulled towards the various planets! See also this animated explanation

### 2.2.2 Conservation of Momentum

From Newton's 2$^{nd}$ and 3$^{r}d$ law we can easily derive the law of conservation of momentum. Assume there are only two point-particle (i.e. particles with no size but with mass), that exert a force on each other. No other forces are present. From N2 we have:

$$\begin{aligned} \frac{d\vec{p}_1}{dt} &= \vec{F}_{21} \\ \frac{d\vec{p}_2}{dt} &= \vec{F}_{12} \end{aligned} \tag{2.66}$$

From N3 we know:

$$\vec{F}_{21} = -\vec{F}_{12} \tag{2.67}$$

And, thus by adding the two momentum equations we get:

$$\left.\begin{array}{ll} \frac{d\vec{p}_1}{dt} & = \vec{F}_{21} \\ \frac{d\vec{p}_2}{dt} & = \vec{F}_{12} = -\vec{F}_{21} \end{array}\right\} \Rightarrow \tag{2.68}$$

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0 \rightarrow \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0 \tag{2.69}$$

$$\Rightarrow \vec{p}_1 + \vec{p}_2 = const\; i.e. doesnotdependontime \tag{2.70}$$

Note the importance of the last conclusion: **if objects interact via a mutual force then the total momentum of the objects cannot change.** No matter what the interaction is. This notion is easily extended to more interacting particles. The crux is that particles interact with one another via forces that obey N3. Thus for three interacting point particles we would have (with $\vec{F}_{ij}$ the force from particle *i* felt by particle *j*):

$$\begin{aligned} \frac{d\vec{p}_1}{dt} &= \vec{F}_{21} + \vec{F}_{31} \\ \frac{d\vec{p}_2}{dt} &= \vec{F}_{12} + \vec{F}_{32} = -\vec{F}_{21} + \vec{F}_{32}\} \\ \frac{d\vec{p}_3}{dt} &= \vec{F}_{13} + \vec{F}_{23} = -\vec{F}_{31} - \vec{F}_{32} \end{aligned} \tag{2.71}$$

Sum these three equations:

$$\begin{aligned} \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} + \frac{d\vec{p}_3}{dt} = 0 \rightarrow \frac{d}{dt}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) = 0 \\ \Rightarrow \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = const.\ i.e.doesnotdependontime \end{aligned} \tag{2.72}$$

For a system of $N$ particles, extension is straight forward.

**Intermezzo: Isaac Newton**

At the end of the year of Galilei's death, Isaac Newton was born in Woolsthorpe-by-Colsterworth in England. He is regarded as the founder of classical mechanics and with that he can be seen as the father of physics.

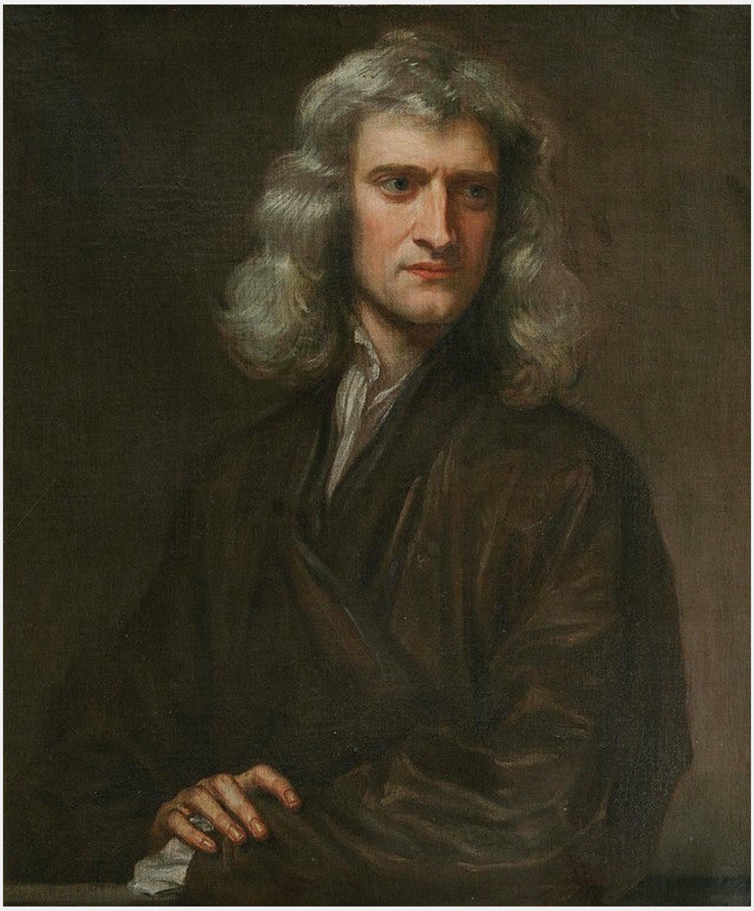


Figure 2.27: Isaac Newton (1642-1727). From Wikimedia Commons, public domain.

In 1661, he started studying at Trinity College, Cambridge. In 1665, the university temporarily closed due to an outbreak of the plague. Newton returned to his home and started working on some of his breakthroughs in calculus, optics and gravitation. Newton's list of discoveries is unsurpassed. He 'invented' calculus (at about the same time and independent of Leibniz). Newton is known for 'the binomium of Newton', the cooling law of Newton. He proposed that light is made of particles and formulated his law of gravity. Finally, he postulated his three laws that started classical mechanics and worked on several ideas towards energy and work. Much of our concepts in physics are based on the early ideas and their subsequent development in classical mechanics. The laws and rules apply to virtually all daily life physical phenomena and only require adaptation when we go to very small scale or extreme velocities and cosmology. In what follows, we will follow his footsteps, but in a modern way that we owe to many physicist and mathematicians that over the years shaped the theory of classical mechanics in a much more comprehensive form. We do not only stand on shoulders of giants, we stand on a platform carried by many.

Interesting to know is that his mentioning of *standing on shoulders* can be interpreted as a sneer towards Robert Hooke (1635-1703), with he was in a fight with over several things. Hooke was a rather short man… See, e.g., Gribbin (2019).

**Important**

In Newtonian mechanics time does not have a preferential direction. That means, in the equations derived based on the three laws of Newton, we can replace $t$ by $-t$ and the motion will have different sign, but that's it. The path/orbit will be the same, but traversed in opposite direction. Also in special relativity this stays the same.

However, in daily life we experience a clear distinction between past, present and future. This difference is not present in this lecture at all. Only by the second of law thermodynamics the time axis obtains a direction, more about this in classes on Statistical Mechanics.

### 2.2.3 Newton's laws applied

#### 2.2.3.1 Force addition, subtraction and decomposition

Newton's laws describe how forces affect motion. Applying them often requires combining multiple forces acting on an object, see Figure 4. This is done through vector addition, subtraction, and decomposition—allowing us to find the net force and to analyze its components in different directions (see this chapter in the book on linear algebra for a full elaboration on vector addition and subtraction).

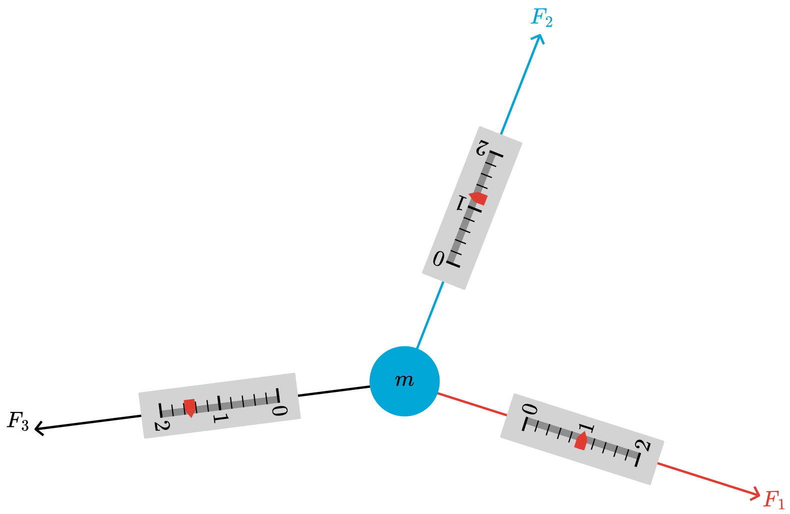



Figure 2.28: Three forces acting on a particle. In which direction will it accelerate?

**Example: Three forces acting on a particle**

Consider three forces acting on a particle:

$\vec{F}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{F}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{F}_3 = \begin{pmatrix} -1 \\ -0.5 \end{pmatrix}$

What is the net force acting on the particle and in which direction will the particle accelerate?

**Exercise 4: Forces acting on a particle in 3D**

Three forces act on a particle with mass $m$:

$$\vec{F}_1 = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}, \vec{F}_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} and \vec{F}_3 = \begin{pmatrix} -1 \\ -0.5 \\ 1 \end{pmatrix} \tag{2.73}$$

Determine the acceleration of this particle.

**Solution to Exercise 4: Forces acting on a particle in 3D**

$$\begin{aligned}\vec{F}_{net} &= \sum \vec{F}_i = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\ &= \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ -0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1-1 \\ 0+1+-0.5 \\ -4+3+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}\end{aligned} \tag{2.74}$$

Hence, the net force acting on the particle is $\sqrt{1^2 + .5^2} = 1.1N$ and the particle will accelerate in the direction $\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$, in essence just like in the previous example. The magnitude of the acceleration is $a = F/m$ and can only be calculated when the mass of the particle is specified.

**Example: Incline**

The box in Figure 5 is at rest. Calculate the frictional force acting on the box.

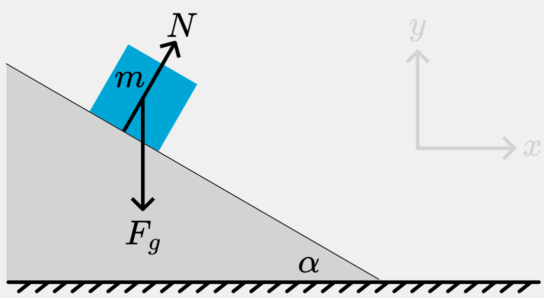



Figure 2.29: A box is at rest on an incline.

**Develop**

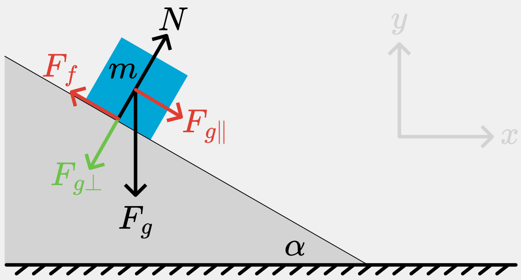
As the box is not moving (i.e. it has a constant velocity) the sum of forces on the box must be equal to zero. In the sketch we see two forces that clearly do not add up to zero. A third force is needed.

**Evaluate**

If we assume that only friction as a third force is present, we require:

$$\sum_i \vec{F}_i = 0 \Rightarrow \vec{F}_g + \overrightarrow{F_N} + \vec{F}_f = 0 \Rightarrow \vec{F}_f = -\vec{F}_g - \vec{F}_N \tag{2.75}$$

We can progress further by assuming that the friction force acts parallel to the slope. With this assumption, we can decompose gravity in its components perpendicular to the slope and parallel to the slope.



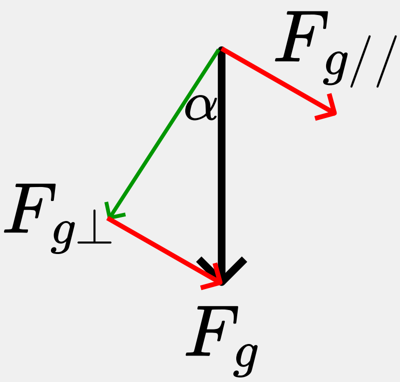

$$\vec{F}_g = \vec{F}_{g//} + \vec{F}_{g\perp} \tag{2.76}$$

The normal force exactly balances the perpendicular component: that is what a normal force does. Friction balances the parallel component of gravity:

$$\vec{F}_f + \vec{F}_{g//} = 0 \rightarrow \vec{F}_f = -\vec{F}_{g//} \tag{2.77}$$

and its magnitude is thus $F_f = F_g \sin \alpha$

**Reminder**




Remember from secondary school how to break down a force vector into components.

#### 2.2.3.2 Acceleration due to gravity

In most cases the forces acting on an object are not constant. However, there is a classical case that is treated in physics (already at secondary school level) where only one, constant force acts and other forces are neglected. Hence, according to Newton's second law, the acceleration is constant.

When we first consider only the motion in the z-direction, we can derive:

$$a = \frac{F}{m} = const. \tag{2.78}$$

Hence, for the velocity:

$$v(t) = v_0 + \int_{t_0}^{t_e} adt = a(t_e - t_0) + v_0 \tag{2.79}$$

assuming $t_0 = 0 and t_e = t \Rightarrow v(t) = v_0 + at$ the position is described by

$$s(t) = \int_0^t v(t)dt = \int_0^t at + v_0 dt = \frac{1}{2}at^2 + v_0 t + s_0 \tag{2.80}$$

Rearranging:

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0 \tag{2.81}$$

**Exercise 5: Tossing a stone in the air** 🌶️

At a height of 1.5 m a stone is tossed in the air with a velocity of 10 m/s.

1. Calculate the maximum height that it reaches.
2. Calculate the time it takes to reach this point.
3. Calculate with which velocity it hits the ground.

**Solution to Exercise 5: Tossing a stone in the air** 🌶️

**Interpret**

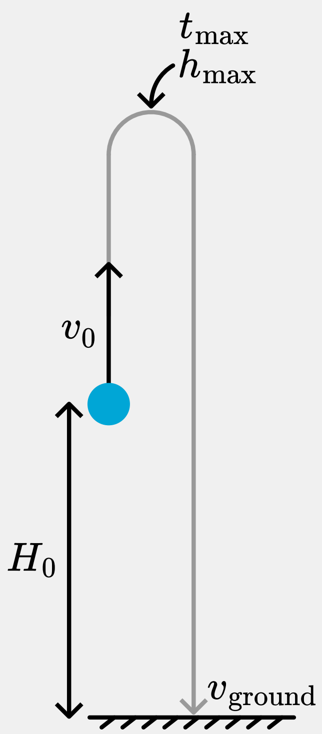



Figure 2.32: A free body diagram of the situation with all relevant quantities.

Only gravity acts on the stone (in the downward direction). We will call the position of the stone at time $t$: $s(t)$
Initial conditions: $t = 0 \rightarrow s(0) = s_0 = 1.5\,m and \dot{s} = v = v_0 = 10$ m/s

**Develop**

1. $s(t) = \frac{1}{2}at^2 + v_0 t + s_0$ Highest point reached when $\dot{s} = 0$
2. $\Delta t = \frac{\Delta v}{a}$
3. $s(t) = \frac{1}{2}at^2 + v_0 t + s_0$. We are interested in the stone hitting the ground. Thus, solve for $s(t) = 0$ to find at what time this happens.

**Evaluate**

1. $\dot{s} = at + v_0 = -gt + v_0 = 0 \Rightarrow t = 1.02s$

$s(1.02) = -\frac{1}{2} * 9.81 * 1.02^2 + 10 * 1.02 + 1.5 = 6.6m$

1. See above.
2. $s(t) = \frac{1}{2}at^2 + v_0 t + s_0 = s_e$

$t = \frac{-v_0 \pm \sqrt{v_0^2 - 4(\frac{1}{2}a(s_0 - s_e))}}{2\frac{1}{2}a} = \frac{-10 \pm \sqrt{10^2 - 4(\frac{1}{2}(-9.81)(1.5))}}{-9.81} = 2.18s$

$v(2.18) = \dot{s}(2.18) = v_0 + at = 10 - 9.81 * 2.18 = -11.3 \text{ m/s}$

Note that $t = -0.14s$ is another solution, but not physically realistic.

**Assess**

The times we calculated are in the right order: First stone is tossed (at $t_0 = 0$), then it reaches its highest point (at $t_m = 1.02$ s). After that it falls and hits the ground at $t_e = 2.18$ s. Thus $t_0 < t_m < t_e$.

Furthermore, the velocity upon impact with the earth is negative as it should: the stone is falling downward. Its magnitude is on the order of the initial upward velocity, which makes sense. Finally, our answers have the right units.

**NOTE:** Some of these solutions can be derived more easily using the concept of *conservation of energy* which will be covered in one of the next chapters.

**Example: 2D-motion**

We only considered motion in the vertical direction, however, objects tend to move in three dimension. We consider now the two-dimensional situation, starting with an object which is horizontally thrown from a height.

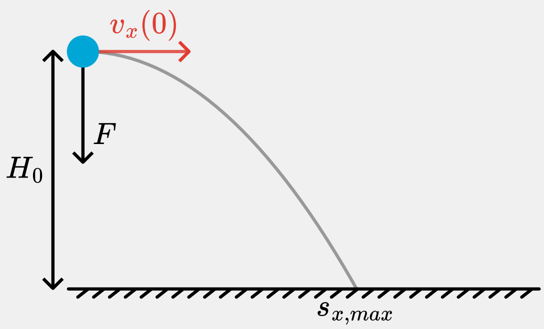



Figure 2.33: A sketch of the situation where an object is thrown horizontally and the horizontal distance should be calculated.

In the situation given in Figure 9 the object is thrown with a horizontal velocity of $v_{x0}$. As no forces in the horizontal direction act on the object (N1), its horizontal motion can be described by

$$s_x(t) = v_{x0}t \tag{2.82}$$

In the vertical direction only the gravitational force acts (N2), hence the motion can be described by (26). Taking the $y$-direction upward, a starting height $y(0) = H_0$ and $v_y(0) = 0$ it becomes:

$$s_y(t) = H_0 - \frac{1}{2}gt^2 \tag{2.83}$$

The total horizontal traveled distance of the object before hitting the ground then becomes:

$$s_{x,max} = v_x\sqrt{\frac{2H_0}{g}} \tag{2.84}$$

This motion is visualized in Figure 10. The trajectory is shown with $s_x$ on the horizontal axis and $s_y$ on the vertical axis. At regular time intervals $\Delta t$, velocity vectors are drawn to illustrate the motion. Note that the horizontal and vertical components of velocity, $v_x$ and $v_y$, vary independently throughout the trajectory. Moreover, $\vec{v}(t)$ is the tangent of $s(t)$.

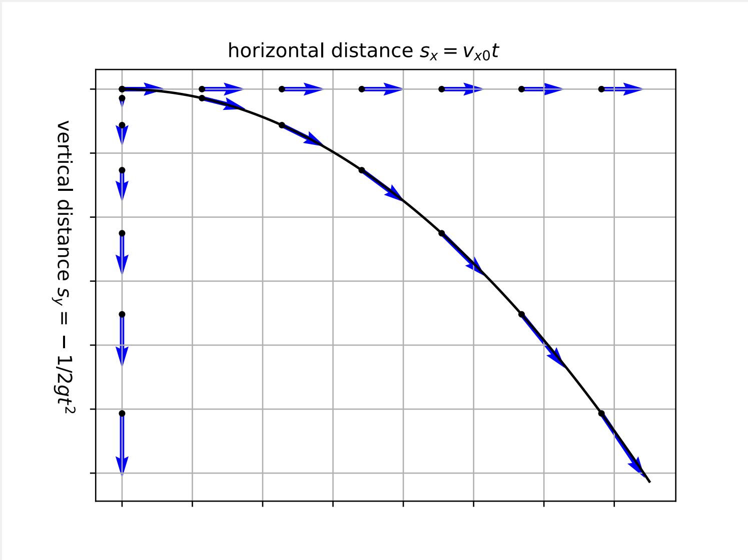



Figure 2.34: The parabolic motion is visualized with blue velocity vectors $v, v_x andv_y$ shown at various points along the trajectory.

**Exercise 6: Horizontal throw** 🌶️

Derive the above expression (29) yourselves.

**Exercise 7: Projectile motion** 🌶️ 🌶️

Watch the recording below. What happens with the horizontal distance traveled per time unit? And with the vertical distance traveled?

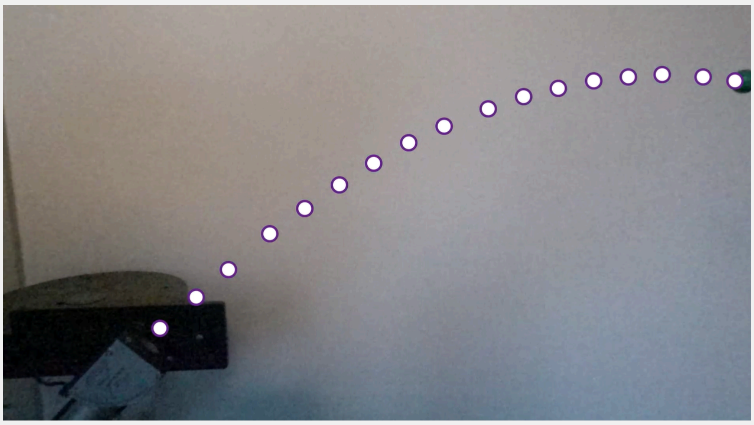


Figure 2.35: A parabolic motion visualized, with the position stored per time unit :alt: A short video of a small ball being shot upward at an angle. For each frame, it position is marked by a dot. The dots make up a parabola.

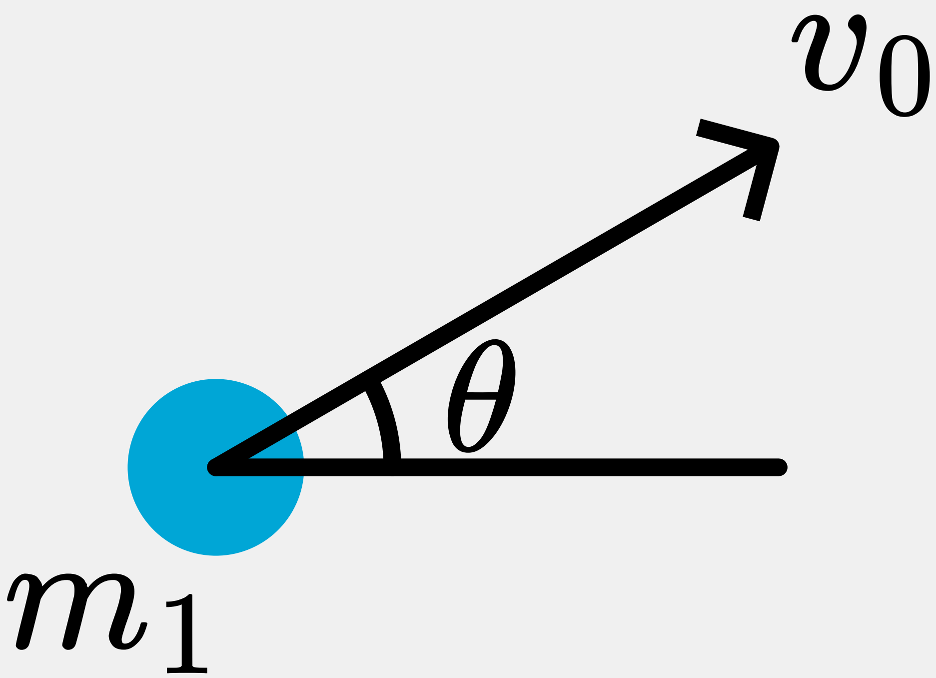
Assume the object with mass $m_1$ is shot from the ground with a velocity of $v_0$ at an angle of $\theta$. Derive where the object hits the ground in terms of $m_1$, $v_0$ and $\theta$.

How does the distance traveled changes when the mass of the object is doubled $m_2 = 2m_1$?

**Solution to Exercise 7: Projectile motion 🌶️ 🌶️**

The horizontal traveled distance is the same per time unit. For the vertical traveled distance it decreases until $v_y = 0$ and then increases.

**Interpret**




**Develop**

The basic formulas are:

$$s_x(t) = v_x t \tag{2.85}$$

and

$$s_y(t) = v_y t - 1/2gt^2 \tag{2.86}$$

**Evaluate**

The horizontal traveled distance is given by:

$$s_x(t) = v_x t = v_0 \cos(\theta) t \tag{2.87}$$

The time the object stays in the air is

$$s_y(t) = v_y t - 1/2gt^2 = 0 \Rightarrow t = 0t = \frac{2v_y}{g} = \frac{2v_0 \sin(\theta)}{g} \tag{2.88}$$

Hence, the maximum distance traveled is:

$$s_x(t) = v_x t = v_0 \cos(\theta) \frac{2v_0 \sin(\theta)}{g} = \frac{2v_o^2 \sin(\theta)\cos(\theta)}{g} \tag{2.89}$$

Note that the distance traveled is independent of the mass!

**Danger**

Understand that the case above is specific in physics: in most realistic contexts multiple forces are acting upon the object. Hence the equation of motion does not become $s(t) = s_0 + v_0 t + 1/2at^2$

**Exercise 8: Constant acceleration due to gravity** 🌶️

We assumed a constant acceleration due to gravity. However, the gravitational force is given by $F = -G\frac{mM}{r^2}$.
Calculate at what height above the earth the acceleration due to gravity has 'significantly' changed from $9.81 \mathrm{m/s^2}$, say to $9.80 \mathrm{m/s^2}$.

**Solution to Exercise 8: Constant acceleration due to gravity** 🌶️

The acceleration of gravity is found by setting the gravitation force equal to $-mg$:

$$-G\frac{mM}{r^2} = -mg(r) \Rightarrow g(r) = G\frac{M}{r^2} \tag{2.90}$$

with $M$ the mass of the earth.

At the surface of the earth, $r = R_e$ we have for the value of $g_e = 9.81 \mathrm{~m/s^2}$. We look for the height above the earth surface where $g$ has dropped to $9.80 \mathrm{~m/s^2}$. If we call this height $H$, we write for the distance to the center of the earth $r = R_e + H$.

Thus, we look for $\frac{g(r)}{g_e} = \frac{9.80(\mathrm{m/s^2})}{9.81(\mathrm{m/s^2})} = 0.999$:

$$\frac{g(r)}{g_e} = \frac{GM/r^2}{GM/R_e^2} \to \frac{R_e^2}{r^2} = \frac{R_e^2}{(R_e + H)^2} = \frac{9.80}{9.81} = 0.999 \tag{2.91}$$

If we solve $H$ from this equation we find: $H = 3.25$ km (we used $R_e = 6378$ km).

**Note**

We could have also looked at the ratios (between $g$ and $r$), and found that $R_2 = \sqrt{.999} \cdot 6378 = 6374.8$ km. Hence, $H = 3.2$ km.

If we would have said: 'significant change' in means $g \to 9.81 \to 9.71 \mathrm{~m/s^2}$, we would have found $H = 32.8$ km.

**Exercise 9: A rocket in space** 🌶️

A rocket moves freely horizontal through space. At position $x = 2$ it turns on its propulsion. At position $x = 4$ it turns off its propulsion. The force due to this propulsion is directed perpendicular to the x-direction.

Provide a sketch of its movement highlighting all important parts.

#### 2.2.3.3 Frictional forces

There are two main types of frictional force:

Exercise 2.37: Particle movement 🌶️ 🌶️

Consider a particle which will travel a distance $x$. Find two different mathematical expressions for a force acting on the particle in such a way that the particle will travel the same distance in the same time for each $F(t)$ compared to a particle which travels at constant speed. Assume no initial velocity for the two particles.

Solution 2.38: Solution to Exercise 1

Uniform motion ($F = ma = 0 \rightarrow s = v_0 t$).

Constant acceleration $a = const \rightarrow s = 1/2at^2$, with $a = \frac{2v_0^2}{s}$.

Consider the third being a harmonic oscillating force field: $F(t) = A\sin(2\pi ft)$ Then the equation of motion becomes:

$$a = F/m = \frac{A}{m}\sin(2\pi ft) \tag{2.92}$$

$$v = \int adt = \frac{A}{m2\pi f}\cos(2\pi ft) + C_0 \tag{2.93}$$

Assuming $v(0) = 0 \rightarrow C_0 = -\frac{A}{m2\pi f}$

And,

$$x = \int vdt = \frac{A}{m(2\pi f)^2}\sin(2\pi ft) + C_0 t + C_1 \tag{2.94}$$

Assuming $x(0) = 0 \rightarrow C_1 = 0$

Hence:

$$x = \frac{A}{m(2\pi f)^2}\sin(2\pi ft) - \frac{A}{m2\pi f}t \tag{2.95}$$

Now, finding traveling the same distance in the same time AND the harmonic oscillation is complete (hence, $f = \frac{1}{t_e}$):

$$v_0 t_e = \frac{At_e^2}{m(2\pi)^2}\sin(2\pi) - \frac{At_e}{m2\pi}t_e \tag{2.96}$$

$$v_0 t_e = -\frac{At_e^2}{m2\pi} \tag{2.97}$$

$$v_0 = -\frac{At_e}{m2\pi} \tag{2.98}$$

$$\frac{m}{A} = -\frac{t_e}{v_{02}\pi} \tag{2.99}$$

- **Static friction** prevents an object from starting to move. It adjusts in magnitude up to a maximum value, depending on how much force is trying to move the object. This maximum is given by

$$F_{static,max} = \mu_s N \tag{2.100}$$

where $\mu_s$ is the coefficient of static friction and $N$ is the normal force. If the applied force exceeds this maximum, the object begins to slide.

- **Kinetic (dynamic) friction** opposes motion once the object is sliding. Its magnitude is generally constant and given by

$$F_{kinetic} = \mu_k N \tag{2.101}$$

where $\mu_k$ is the coefficient of kinetic friction. This force does not depend on the velocity of the object, only on the normal force and surface characteristics.

Friction always acts opposite to the direction of intended or actual motion and is essential in both preventing and controlling movement.

| Material Pair | Static Friction ($\mu_s$) | Kinetic Friction ($\mu_k$) |
|---|---|---|
| Rubber on dry concrete | 1.0 | 0.8 |
| Steel on steel (dry) | 0.74 | 0.57 |
| Wood on wood (dry) | 0.5 | 0.3 |
| Aluminum on steel | 0.61 | 0.47 |
| Ice on ice | 0.1 | 0.03 |
| Glass on glass | 0.94 | 0.4 |
| Copper on steel | 0.53 | 0.36 |
| Teflon on Teflon | 0.04 | 0.04 |
| Rubber on wet concrete | 0.6 | 0.5 |
| Leather on wood | 0.56 | 0.4 |

*Values are approximate and can vary depending on surface conditions.*

**Note**

Not always are the friction coefficients constants. They may, for instance, depend on the relative velocity between the two materials.

#### 2.2.3.4 Momentum example

The above theoretical concept is simple in its ideas:

- a particle changes its momentum whenever a force acts on it;
- momentum is conserved;
- action = - reaction.

But it is incredible powerful and so generic, that finding when and how to use it is much less straight forward. The beauty of physics is its relatively small set of fundamental laws. The difficulty of physics is these laws can be applied to almost anything. The trick is how to do that, how to start and get the machinery running. That can be very hard. Luckily there is a recipe to master it: it is called practice.

Exercise 2.39: Block on an incline

A block with mass $m$ is put on an inclined plane of which we can change the inclination angle $\theta$.

1. Determine the angle at which it starts to slide in terms of mass $m$, inclination angle $\theta$, acceleration due to gravity $g$ and coefficient of static friction $\mu_s$.
2. Once it starts to slide, it will accelerate. Determine its acceleration in terms of mass $m$, inclination angle $\theta$, acceleration due to gravity $g$ and coefficient of kinetic friction $\mu_f$.

Solution 2.40: Solution to Exercise 2

1. There a two forces acting on $m$ parallel to the inclined plane: friction and gravity's component parallel to the slope. These two determine the motion along the slope: if we tilt the plane the component of gravity parallel to the slope gets bigger. The particle will start moving when we pass: $F_{g_x} = F_s \to mg\sin(\theta) = mg\mu_s\cos(\theta) \Rightarrow \theta_{max} = \tan^{-1}(\mu_s)$
2. Once the particle is sliding downward, gravity and the kinetic friction determine how fast:

$$F_{net} = F_{g_x} - F_f \to ma = mg\sin(\theta) - mg\mu_k\cos(\theta) \Rightarrow \tag{2.102}$$

and

$$a = g(\sin(\theta) - \mu_k\cos(\theta)) \tag{2.103}$$

Exercise 2.41: 🌶️

A point particle (mass $m$) is dropped from rest at a height $h$ above the ground. Only gravity acts on the particle with a constant acceleration $g$ ($= 9.813\ \mathrm{m/s^2}$).

- Find the momentum when the particle hits the ground.
- What would be the earth' velocity upon impact?

### 2.2.4 Forces & Inertia

Newton's laws introduce the concept of force. Forces have distinct features:

- forces are vectors, that is, they have magnitude and direction;
- forces change the motion of an object:
  - they change the velocity, i.e. they accelerate the object

$$\vec{a} = \frac{\vec{F}}{m} \leftrightarrow d\vec{v} = \vec{a}dt = \frac{\vec{F}dt}{m} \tag{2.108}$$

- or, equally true, they change the momentum of an object

$$\frac{d\vec{p}}{dt} = \vec{F} \leftrightarrow d\vec{p} = \vec{F}dt \tag{2.109}$$

Many physicists like the second bullet: forces change the momentum of an object, but for that they need time to act.

Momentum is a more fundamental concept in physics than acceleration. That is another reason why physicists prefer the second way of looking at forces.

**Connecting physics and calculus**

Let's look at a particle of mass $m$, that has initially (say at $t = 0$) a velocity $v_0$. For $t > 0$ the particle is subject to a force that is of the form $F = -bv$. This is a kind of frictional force: the faster the particle goes, the larger the opposing force will be.

We would like to know how the position of the particle is as a function of time.

We can answer this question by applying Newton 2:

$$m\frac{dv}{dt} = F \Rightarrow m\frac{dv}{dt} + bv = 0 \tag{2.110}$$

Clearly, we have to solve a differential equation which states that if you take the derivative of $v$ you should get something like $-v$ back. From calculus we know, that

Solution 2.42: Solution to Exercise 3

Let's do this one together. We follow the standard approach of IDEA: Interpret (and make your sketch!), develop (think 'model'), evaluate (solve your model) and assess (does it make any sense?).

**Interpret**

First a sketch: draw what is needed, no more, no less.

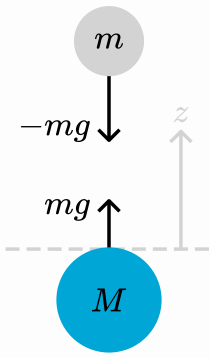



Figure 2.43: align: center

**Develop**

Actually this is half of the work, as when deciding what is needed we need to think what the problem really is. Above, is a sketch that could work. Both the object $m$ and the earth (mass $M$) are drawn schematically. On each of them acts a force, where we know that on $m$ standard gravity works. As a consequence of N3, a force equal in strength but opposite in direction acts on $M$.
Why do we draw forces? Well, the question mentions 'momentum the particle hits the ground'. Momentum and forces are coupled via N2.

We have drawn a z-coordinate: might be handy to remind us that this looks like a 1D problem (remember: momentum and force are both vectors).

As a first step, we ignore the motion of the earth. Argument? The magnitude of the ratio of the acceleration of earth over object is given by:

$$\frac{a_e}{a_o} = \frac{\mid F_{o \to e} \mid / m_e}{\mid F_{e \to o} \mid / m_o} = \frac{m_o}{m_e} \tag{2.104}$$

here for the second equality we used N3.

For all practical purposes, the mass of the object is many orders of magnitude smaller than that of the earth. Hence, we can conclude that the acceleration of the earth is many orders of magnitude less than that of the object. The latter is of the order of $g$, gravity's acceleration constant at the earth. Thus, the acceleration of the earth is next to zero and we can safely assume our lab system, that is connected to the earth, can be treated as an inertial system with, for us, zero velocity.

**Evaluate**

The remainder is straightforward. Now we have an object, that moves under a constant force. So its velocity will increase linearly in time:

$$\frac{dp}{dt} = -mg \Rightarrow p(t) = m\underbrace{v_0}_{=0} - mgt = -mgt \tag{2.105}$$

With the momentum we can calculate the velocity and from the velocity the position:

$$v = -gt \Rightarrow \frac{dz}{dt} = -gt \Rightarrow z(t) = z_0 - \frac{1}{2}gt^2 = H - \frac{1}{2}gt^2 \tag{2.106}$$

**Assess**

We found that the particle changed its momentum from $p_i = 0$ to $p_f = -mv$. The earth compensates for this, to keep momentum conserved. That gave that earth got a tiny, tiny upwards velocity. We could estimate the displacement of the earth. Suppose, the particle has mass $m$=1kg and is dropped from a height $H$ = 100m.

exponential function have the feature that when we differentiate them, we get them back. So, we will try $v(t) = Ae^{-\mu t}$ with $A$ and $\mu$ to be determined constants.

We substitute our trial $v$:

$$m \cdot A \cdot -\mu e^{-\mu t} + b \cdot Ae^{-\mu t} = 0 \tag{2.111}$$

This should hold for all $t$. Luckily, we can scratch out the term $e^{-\mu t}$, leaving us with:

$$-mA\mu + Ab = 0 \tag{2.112}$$

We see, that also our unknown constant $A$ drops out. And, thus, we find

$$\mu = \frac{b}{m} \tag{2.113}$$

Next we need to find $A$: for that we need an initial condition, which we have: at $t = 0$ is $v = v_0$. So, we know:

$$v(t) = Ae^{-\frac{b}{m}t} and v(0) = v_0 \tag{2.114}$$

From the above we see: $A = v_0$ and our final solution is:

$$v(t) = v_0 e^{-\frac{b}{m}t} \tag{2.115}$$

From the solution for $v$, we easily find the position of $m$ as a function of time. Let's assume that the particle was in the origin at $t = 0$, thus $x(0) = 0$. So, we find for the position

$$\frac{dx}{dt} \equiv v = v_0 e^{-\frac{b}{m}t} \Rightarrow x = v_0 \cdot \left(-\frac{m}{b} e^{-\frac{b}{m}t}\right) + B \tag{2.116}$$

We find $B$ with the initial condition and get as final solution:

$$x(t) = \frac{mv_0}{b}\left(1 - e^{-\frac{b}{m}t}\right) \tag{2.117}$$

If we inspect and assess our solution, we see: the particle slows down (as is to be expected with a frictional force acting on it) and eventually comes to a stand still. At that moment, the force has also decreased to zero, so the particle will stay put.

#### 2.2.4.1 Inertia

Inertia is denoted by the letter $m$ for mass. And mass is that property of an object that characterizes its resistance to changing its velocity. Actually, we should have written something like $m_i$, with subscript i denoting inertia.

Why? There is another property of objects, also called mass, that is part of Newton's Gravitational Law.

Two bodies of mass $m_1$ and $m_2$ that are separated by a distance $r_{12}$ attract each other via the so-called gravitational force ($\hat{r}_{12}$ is a unit vector along the line connecting $m_1$ and $m_2$):

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r_{12}^2}\hat{r}_{12} \tag{2.118}$$

Here, we should have used a different symbol, rather than $m$. Something like $m_g$, as it is by no means obvious that the two 'masses' $m_i$ and $m_g$ refer to the same property. If you find that confusing, think about inertia and electric forces. Two particles with each an electric charge, $q_1$ and $q_2$, respectively exert a force on each other known as the Coulomb force:

$$\vec{F}_{C,12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12} \tag{2.119}$$

We denote the property associated with electric forces by $q$ and call it charge. We have no problem writing

$$\begin{aligned} \vec{F} &= m\vec{a} \\ \vec{F}_C &= \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r} \end{aligned} \tag{2.120}$$

We do not confuse $q$ by $m$ or vice versa. They are really different quantities: $q$ tells us that the particle has a property we call 'charge' and that it will respond to other charges, either being attracted to, or repelled from. How fast it will respond to this force of another charged particle depends on $m$. If $m$ is big, the particle will only get a small acceleration; the strength of the force does not depend on $m$ at all. So far, so good. But what about $m_g$? That property of a particle that makes it being attracted to another particle with this same property, that we could have called 'gravitational charge'. It is clearly different from 'electrical charge'. But would it have been logical that it was also different from the property inertial mass, $m_i$?

$$\begin{aligned} \vec{F} &= m_i\vec{a} \\ \vec{F}_g &= -G\frac{m_g M_g}{r^2}\hat{r} \end{aligned} \tag{2.121}$$

As far as we can tell (via experiments) $m_i$ and $m_g$ are the same. Actually, it was Einstein who postulated that the two are referring to the same property of an object: there is no difference.

**Force field**
We have seen, forces like gravity and electrostatics act between objects. When you push a car, the force is applied locally, through direct contact. In contrast, gravitational and electrostatic forces act over a distance — they are present throughout space, though they still depend on the positions of the objects involved.

One powerful way to describe how a force acts at different locations in space is through the concept of a **force field**. A force field assigns a force vector (indicating both direction and magnitude) to every point in space, telling you what force an object would experience if placed there.

For example, the graph below at the left shows a gravitational field, described by $\vec{F}_g = G\frac{mM}{r^2}\hat{r}$. Any object entering this field is attracted toward the central mass with a force that depends on its distance from that mass's center.

The figure on the right shows the force field that a positively charged particle would feel due to the presence of 2 negatively charged particles (both of the same charge). Clearly this is a much more complicated force field.

**Measuring mass or force**
So far we did not address how to measure force. Neither did we discuss how to measure mass. This is less trivial than it looks at first side. Obviously, force and mass are coupled via N2: $F = ma$.

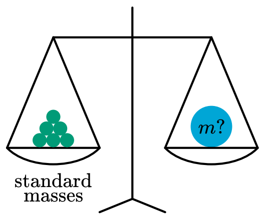



Figure 2.44: Can force be measured using a balance?

The acceleration can be measured when we have a ruler and a clock, i.e. once we have established how to measure distance and how to measure time intervals, we can measure position as a function of time and from that velocity and acceleration.

But how to find mass? We could agree upon a unit mass, an object that represents by definition 1kg. In fact we did. But that is only step one. The next question is: how do we compare an unknown mass to our standard. A first reaction might be: put them on a balance and see how many standard kilograms you need (including fractions of it) to balance the unknown mass. Sounds like a good idea, but is it? Unfortunately, the answer is not a ‘yes’.

As on second thought: the balance compares the pull of gravity. Hence, it ‘measures’ gravitational mass, rather than inertia. Luckily, Newton’s laws help. Suppose we let two objects, our standard mass and the unknown one, interact under their mutual interaction force. Every other force is excluded. Then, on account on N2 we have

$$\begin{cases} m_1 a_1 = F_{21} \\ m_2 a_2 = F_{12} = -F_{21} \end{cases} \tag{2.122}$$

where we used N3 for the last equality. Clearly, if we take the ratio of these two equations we get:

$$\frac{m_1}{m_2} = \left| \frac{a_2}{a_1} \right| \tag{2.123}$$

irrespective of the strength or nature of the forces involved. We can measure acceleration and thus with this rule express the unknown mass in terms of our standard.

**Note**

We will not use this method to measure mass. We came to the conclusion that we can’t find any difference in the gravitational mass and the inertial mass. Hence, we can use scales and balances for all practical purposes. But the above shows, that we can safely work with inertial mass: we have the means to measure it and compare it to our standard kilogram.

Now that we know how to determine mass, we also have solved the problem of measuring force. We just measure the mass and the acceleration of an object and from N2 we can find the force. This allows us to develop ‘force measuring equipment’ that we can calibrate using the method discussed above.

**Intermezzo: kilogram, unit of mass**

In 1795 it was decided that 1 gram is the mass of 1 $cm^3$ of water at its melting point. Later on, the kilogram became the unit for mass. In 1799, the *kilogramme des Archives* was made, being from then on the prototype of the unit of mass. It has a mass equal to that of 1 liter of water at 4°C (when water has its maximum density).

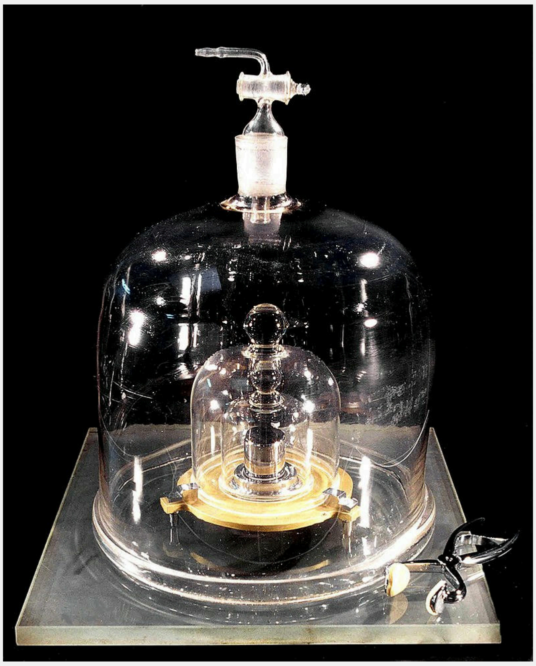


Figure 2.45: The International Prototype of the Kilogram, whose mass was defined to be one kilogram from 1889 to 2019. Picture by BIPM, CC BY-SA 3.0 igo, https://commons.wikimedia.org/w/index.php?curid=117707466

In recent years, it became clear that using such a standard kilogram does not allow for high precision: the mass of the standard kilogram was, measured over a long time, changing. Not by much (on the order of 50 micrograms), but sufficient to hamper high precision measurements and setting of other standards. In modern physics, the kilogram is now defined in terms of Planck's constant. As Planck's constant has been set (in 2019) at exactly $h = 6.62607015 \cdot 10^{-34} kgm^2s^{-1}$, the kilogram is now defined via $h$, the meter and second.

#### 2.2.4.2 Eötvös experiment on mass

The question whether inertial mass and gravitational mass are the same has put experimentalists to work. It is by no means an easy question. Gravity is a very weak force. Moreover, determining that two properties are identical via an experiment is virtually impossible due to experimental uncertainty. Experimentalist can only tell the outcome is 'identical' within a margin. Newton already tried to establish experimentally that the two forms of mass are the same. However, in his days the inaccuracy of experiments was rather large. Dutch scientist Simon Stevin concluded in 1585 that the difference must be less than 5%. He used his famous 'drop masses from the church' experiments for this (they were primarily done to show that every mass falls with the same acceleration).

A couple of years later, Galilei used both fall experiments and pendula to improve this to: less than 2%. In 1686, Newton using pendula managed to bring it down to less than 1‰ .

An important step forward was set by the Hungarian physicist, Loránd Eötvös (1848-1918). We will here briefly introduce the experiment. For a full analysis, we need knowledge about angular momentum and centrifugal forces that we do not deal with in this book.

**The experiment**

The essence of the Eötvös experiment is finding a set up in which both gravity (sensitive to the gravitational mass) and some inertial force (sensitive to the inertial mass) are present. Obviously, gravitational forces between two objects out of our daily life are extremely small. These will be very difficult to detect and thus introduce a large error if the experiment relies on measuring them. Eötvös came up with a different idea. He connected two different objects with different masses, $m_1$ and $m_2$, via a (almost) massless rod. Then, he attached a thin wire to the rod and let it hang down.

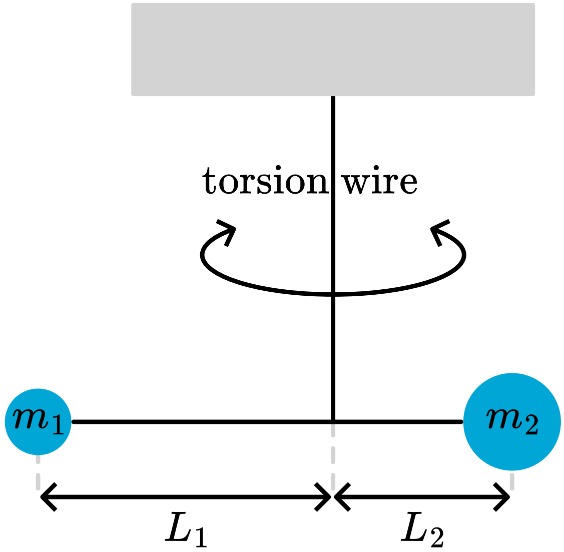



Figure 2.46: align: center :alt: Mass m1 and m2 are connected to either end of a horizontal rod. The rod, in turn, is connected by a vertical wire to the ceiling. The rod can rotate around its suspension point.

Torsion balance used by Eötvös.

This is a sensitive device: any mismatch in forces or torques will have the setup either tilt or rotate a bit. Eötvös attached a tiny mirror to one of the arms of the rod. If you shine a light beam on the mirror and let it reflect and be projected on a wall, then the smallest deviation in position will be amplified to create a large motion of the light spot on the wall.

In Eötvös experiment two forces are acting on each of the masses: gravity, proportional to $m_g$, but also the centrifugal force $F_c = m_i R\omega^2$, the centrifugal force stemming from the fact that the experiment is done in a frame of reference rotating with the earth. This force is proportional to the inertial mass. The experiment is designed such that if the rod does not show any rotation around the vertical axis, then the gravitational mass and inertial mass must be equal. It can be done with great precision and Eötvös observed no measurable rotation of the rod. From this he could conclude that the ratio of the gravitational over inertial mass differed less from 1 than $5 \cdot 10^{-8}$. Currently, experimentalist have brought this down to $1 \cdot 10^{-15}$.

**Note**

The question is not if $m_i/m_g$ is different from 1. If that was the case but the ratio would always be the same, then we would just rescale $m_g$, that is redefine the value of the gravitational const $G$ to make $m_g$ equal to $m_i$. No, the question is whether these two properties are separate things, like mass and charge. We can have two objects with the same inertial mass but give them very different charges. In analogy: if $m_i$ and $m_g$ are fundamentally different quantities then we could do the same but now with inertial and gravitational mass.

**Tip**

Want to know more about this experiment? Watch this videoclip.

### 2.2.5 Examples, exercises and solutions

Updated: 18 jan 2026 Here are some examples and exercises that deal with forces. Make sure you practice IDEA.

#### 2.2.5.1 Exercises set 1

**Exercise 1: Force on a particle** 🌶️

Consider a point particle of mass $m$, moving at a velocity $v_0$ along the x-axis. At $t = 0$ a constant force acts on the particle in the negative x-direction. The force lasts for a small time interval $\Delta t$.

What is the strength of the force, if it brings the particle exactly to a zero-velocity? Start by making a drawing.

**Exercise 2: Shooting a ball** 🌶️

A ball is shot from a 10m high hill with a velocity of 10m/s under an angle of $30^o$, see Figure 1.

1. How long is the ball in the air?
2. How far does it travel in the horizontal direction?
3. With what velocity does the ball hit the ground?

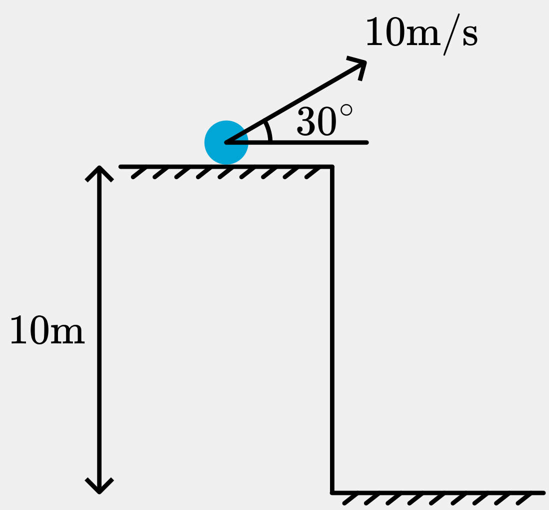



Figure 2.47: A ball on a hill launched under an angle.

**Exercise 3: Constant force on a particle** 🌶️

A particle of mass $m$ moves along the $x$-axis. At time $t = 0$ it is at the origin with velocity $v_0$. For $t > 0$, a constant force acts on the particle. This is a 1-dimensional problem.

- Derive the acceleration of the particle as a function of time.
- Derive the velocity of the particle as a function of time.
- Derive the position of the particle as a function of time.

**Exercise 4: Time dependent force on a particle** 🌶️

A particle of mass $m$ moves along the $x$-axis. At time $t = 0$ it is at the origin with velocity $v_0$. For $t > 0$ the particle is subject to a force $F_0 \sin(2\pi f_0 t)$. This is a 1-dimensional problem.

- Calculate the acceleration of the particle as a function of time.
- Calculate the velocity of the particle as a function of time.

- Calculate the position of the particle as a function of time.

**Exercise 5: Particle trajectory** 🌶️

A particle follows a straight path with a constant velocity. At $t = 0$ the particle is at point $A$ with coordinate $(0, y_A)$, while at $t = t_1$ it is at $B$ with coordinate $(x_B, 0)$. The coordinates are given in a Cartesian system. The problem is 2-dimensional.

1. Make a sketch.
2. Find the position of the particle at arbitrary time $0 < t < t_1$.
3. Derive the velocity of the particle from position as function of time.

Represent vectors in a Cartesian coordinate system using the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

**Exercise 6: Different coordinate systems** 🌶️ 🌶️

In Classical Mechanics we often use a coordinate system to describe motion of object. In this exercise, you will look at two Cartesian coordinate systems. System S has coordinates $(x, y)$ and corresponding unit vectors $\hat{x}$ and $\hat{y}$.
The second system, S', uses $(x', y')$ and corresponding unit vectors. The $x'$-axis makes an angle of 30° with the $x$-axis (measured counter-clockwise).

1. Make a sketch.
2. Determine the relations between $\hat{x}'$ and $\hat{x}, \hat{y}$ as well as between $\hat{y}'$ and $\hat{x}, \hat{y}$

An object has, according to S, a velocity of $\vec{v} = 3\hat{x} + 5\hat{y}$.
1. Determine the velocity according to S'.

**Exercise 7: Rotating unit vectors** 🌶️

According to your observations, a particle is located at position (1,0) (you use a Cartesian coordinate system). The particle has no velocity and no forces are acting on it.
Another observer, S', uses a Cartesian coordinate system described by $(x', y')$. You notice that her unit vectors rotate at a constant speed compared to your unit vectors:

$$\hat{x}' = \cos(2\pi f t)\hat{x} + \sin(2\pi f t)\hat{y} \tag{2.124}$$

$$\hat{y}' = -\sin(2\pi f t)\hat{x} + \cos(2\pi f t)\hat{y} \tag{2.125}$$

1. Find the position of the particle according to the other observer, S'.
2. Calculate the velocity of the particle according to S'.

**Exercise 8: Moving over a frictionless table** 🌶️

A particle of mass $m$ moves at a constant velocity $v_0$ over a frictionless table. The direction it is moving in, is at 45° with the positive $x$-axis. At some point in time, the particle experiences a force $\vec{F} = -b\vec{v}$ with $b > 0$.
We call this time $t = 0$ and take the position of the particle at that time as our origin.

1. Make a sketch.
2. Determine whether this problem needs to be analyzed as a 1D or a 2D problem.
3. Set up N2 in the form $m\frac{d\vec{v}}{dt} = \vec{F}$
4. Solve N2 and find the velocity of the particle as a function of time.
5. What happens to the particle for large $t$?

**Exercise 9: Parabolic trajectory with maximum area**[3] 🌶️ 🌶️

[3]Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

A ball is thrown at speed $v$ from zero height on level ground. We want to find the angle $\theta$ at which it should be thrown so that the area under the trajectory is maximized.

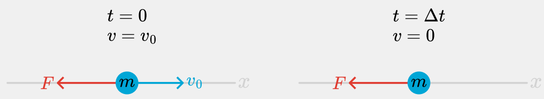
1. Sketch of the trajectory of the ball.
2. Use dimensional analysis to relate the area to the initial speed $v$ and the gravitational acceleration $g$.
3. Write down the $x$ and $y$ coordinates of the ball as a function of time.
4. Find the total time the ball is in the air.
5. The area under the trajectory is given by $A = \int y \mathrm{d}x$. Make a variable transformation to express this integral as an integration over time.
6. Evaluate the integral. Your answer should be a function of the initial speed $v$ and angle $\theta$.
7. From your answer at (6), find the angle that maximizes the area, and the value of that maximum area.

**Exercise 10: Two attracting particles[4]**

Two particles on a line are mutually attracted by a force $F = -ar$, where $a$ is a constant and $r$ the distance of separation. At time $t = 0$, particle A of mass $m$ is located at the origin, and particle B of mass $m/4$ is located at $r = 5.0$ cm. Both particles have zero velocity at $t = 0$. If the particles are at rest at $t = 0$, at what value of $r$ do they collide?

#### 2.2.5.2 Answers set 1

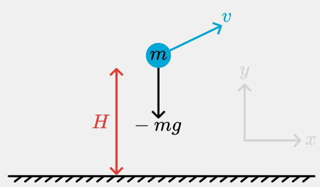
**Solution to Exercise 1: Force on a particle**




$\vec{F} = -\frac{m v_0}{\Delta t} \hat{x}$

**Solution to Exercise 2: Shooting a ball**

**Interpret**




**Develop**

We know $v_y = v \sin(\theta)$ and $v_x = v \cos(\theta)$.

The motion of the ball can be spilt in two components: horizontal, i.e. x-direction, and vertical, tha is y-direction.

In the vertical direction gravity acts: $F_y = -mg$. Thus the equation of motion in the y-direction is: $ma_y = F_y = -mg.Theverticalpositioncanthusbeexpressedas$ s_y(t)=s_{y0}+v_{y0}t-1/2gt^2$.

In the horizontal direction no force is active, thus: $ma_x = 0 \rightarrow s_x(t) = s_{x0} + v_{x0} t$

[4]Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

The magnitude of the velocity of the ball hitting the ground can be expressed in terms of $v_x$ and $v_y$ as $v_e = \sqrt{v_x^2 + v_y^2}$

**Evaluate**

We have as initial velocity: $v_{y0} = v\sin(\theta) = 10 * \sin(30) = 5m/s$
$v_{x0} = v\cos(\theta) = 10 * \cos(30) = 5\sqrt{3}m/s$

Solving $s_y(t) = s_{y0} + v_{y0}t - 1/2gt^2$ for $s_y = 0$ with $s_{y0} = H$ gives for the time the ball is in the air:

$$t_{air} = \frac{v_{y0}}{g} + \sqrt{\frac{v_{y0}^2}{g^2} + \frac{2H}{g}} = 2.77s \tag{2.126}$$

Next, we realize that $v_x = const = v_{x0}$ as there is no force acting in the x-direction. Thus the horizontal distance traveled is

$$\Delta x = v_{X0} t_{air} = 24.0 \text{m}

For the velocity when hitting the ground is (that is, its magnitude), we need both the x and y-component:

$$\begin{aligned} v_x &= v_{x0} = 8.66m/s \\ v_y &= v_{0y} - gt \rightarrow v_y(t_{air}) = \sqrt{v_{y0}^2 + 2gH} = 14.9m/s \end{aligned} \tag{2.127}$$

$$v_{ground} = \sqrt{v_x^2 + v_y^2} = \sqrt{v_{x0}^2 + v_y^2} = 17.2m/s \tag{2.128}$$

**Assess**

The velocity upon impact is larger than the initial velocity. This makes sense. The ball first travels upwards, then downwards and will pass $s_y = H$ again on the downward motion. Then it will further accelerate to the ground and thus have a larger y-component of the velocity than at the start.

## Solution to Exercise 3: Constant force on a particle 🌶️

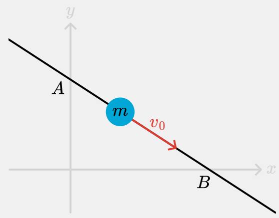
1. $a = \frac{F}{m}$ is constant
2. $v(t) = v_0 + at$
3. $x(t) = v_0 t + \frac{1}{2}at^2$

## Solution to Exercise 4: Time dependent force on a particle 🌶️

1. $a = \frac{F}{m} = \frac{F_0}{m}\sin(2\pi f_0 t)$ is **not** constant
2. $v(t) = v_0 + \frac{F_0}{2\pi f_0 m}(1 - \cos(2\pi f_0 t))$
3. $x(t) = v_0 t + \frac{F_0}{2\pi f_0 m}t - \frac{F_0}{4\pi^2 f_0^2 m}\sin 2\pi f_0 t$

## Solution to Exercise 5: Particle trajectory 🌶️

1.




2. Particle moves at constant velocity, thus path is a straight line:

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t = x_0\hat{\imath} + y_0\hat{\jmath} + v_{0x}t\hat{\imath} + v_{0y}t\hat{\jmath} \tag{2.129}$$

At $t = 0 : \vec{r}(0) = 0\hat{\imath} + y_A\hat{\jmath} \to \vec{r}(0) = \vec{r}_0 = 0\hat{\imath} + y_A\hat{\jmath} \to x_0 = 0 and y_0 = y_A$

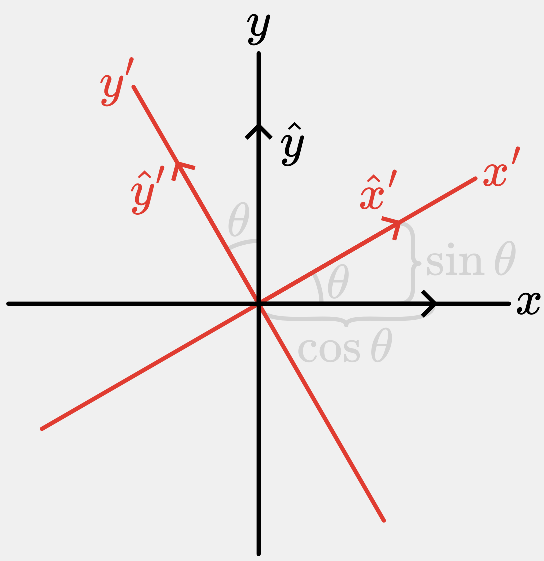
At $t = t_1$:

$$\begin{aligned}\vec{r}(t_1) &= x_B\hat{\imath} + 0\widehat{()}j \to \\ \vec{r}(t_1) &= \vec{r}_0 + \vec{v}_0 t_1 \\ &= (0 + v_{0x}t_1)\hat{\imath} + (y_A + v_{0y}t_1)\hat{\jmath} \to \\ v_{0x} &= \frac{x_B}{t_1} and v_{0y} = -\frac{y_A}{t_1}\end{aligned} \tag{2.130}$$

3. Thus, we find $\vec{v} = \frac{x_B}{t_1}\hat{\imath} - \frac{y_A}{t_1}\hat{\jmath}$

**Solution to Exercise 6: Different coordinate systems 🌶️ 🌶️**

1.




2.

$$\begin{aligned}\hat{x}' &= \cos\theta\hat{x} + \sin\theta\hat{y} = \frac{1}{2}\sqrt{3}\hat{x} + \frac{1}{2}\hat{y} \\ \hat{y}' &= -\sin\theta\hat{x} + \cos\theta\hat{y} = -\frac{1}{2}\hat{x} + \frac{1}{2}\sqrt{3}\hat{y}\end{aligned} \tag{2.131}$$

2. Invert:

$$\begin{aligned}\hat{x} &= \cos\theta\hat{x}' - \sin\theta\hat{y}' = \frac{1}{2}\sqrt{3}\hat{x}' - \frac{1}{2}\hat{y}' \\ \hat{y} &= \sin\theta\hat{x}' + \cos\theta\hat{y}' = \frac{1}{2}\hat{x}' + \frac{1}{2}\sqrt{3}\hat{y}'\end{aligned} \tag{2.132}$$

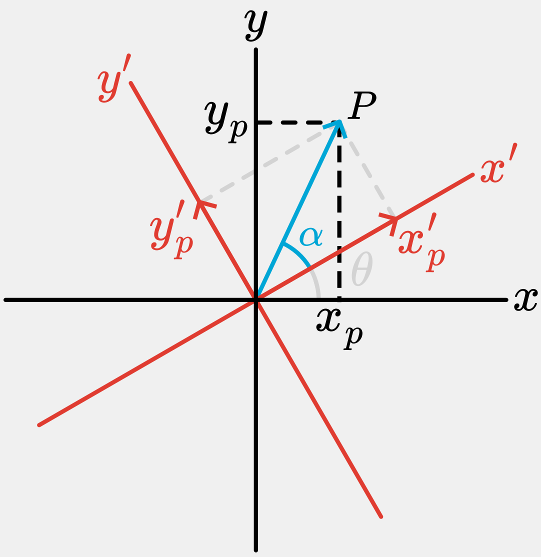
velocity:

$$\begin{aligned}\vec{v} &= v_x\hat{x} + v_y\hat{y} \\ &= v_x(\cos\theta\hat{x}' - \sin\theta\hat{y}') + v_y(\sin\theta\hat{x}' + \cos\theta\hat{y}') \\ &= (v_x\cos\theta + v_y\sin\theta)\hat{x}' + (-v_x\sin\theta + v_y\cos\theta)\hat{y}'\end{aligned} \tag{2.133}$$

from which we find

$$\vec{v} = \left(\frac{3}{2}\sqrt{3} + \frac{5}{2}\right)\hat{x}' + \left(-\frac{3}{2} + \frac{5}{2}\sqrt{3}\right)\hat{y}' \tag{2.134}$$

**Solution to Exercise 7: Rotating unit vectors** 🌶️




$$\begin{aligned}\hat{x}' &= \cos(2\pi ft)\hat{x} + \sin(2\pi ft)\hat{y} \\ \hat{y}' &= -\sin(2\pi ft)\hat{x} + \cos(2\pi ft)\hat{y}\end{aligned} \tag{2.135}$$

The unit vectors of S' rotate with a frequency $f$ with respect to the unit vectors of S. This means, that the coordinate system of S' rotates: the rotation angle is a function of time, i.e. $\theta(t) = 2\pi ft$

From the figure we see, that the coordinates of a point P, $(x_p, y_p)$ according to S, are related to those used by S', $(x'_p, y'_p)$ via:

$$\begin{aligned}x_p &= OP\cos(\alpha+\theta) = OP(\cos\alpha\cos\theta - \sin\alpha\sin\theta) = x'_p\cos\theta - y'_p\sin\theta \\ y_p &= OP\sin(\alpha+\theta) = OP(\cos\alpha\sin\theta + \sin\alpha\cos\theta) = x'_p\sin\theta + y'_p\cos\theta\end{aligned} \tag{2.136}$$

or written as the coordinate transformation:

$$\begin{aligned}x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta\end{aligned} \tag{2.137}$$

with its inverse

$$\begin{aligned}x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta\end{aligned} \tag{2.138}$$

Note that in this case $\theta = 2\pi ft$, that is: it is a function of $t$.

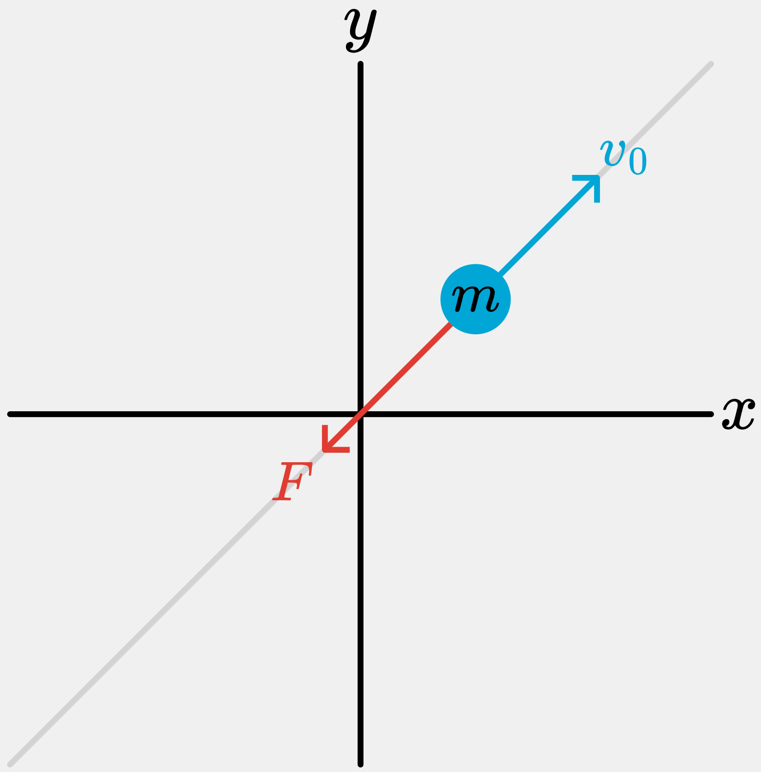
a) From the above relation we find that the point (1,0) in S will be denoted by S' as $(x'(t), y'(t)) = (\cos(2\pi ft), -\sin(2\pi ft))$

b) the velocity of the point (1,0) in S is according to S of course zero: $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$ S' will say:

$$x'(t) = \cos(2\pi f t) \rightarrow \frac{dx'}{dt} = -2\pi f \sin(2\pi f t)$$
$$y'(t) = -\sin(2\pi f t) \rightarrow \frac{dy'}{dt} = 2\pi f \cos(2\pi f t) \tag{2.139}$$

**Solution to Exercise 8: Moving over a frictionless table** 🌶️

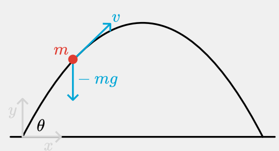
1.




2. Since $\vec{v}_0$ and $\vec{F}$ are parallel, the particle will not deviate from the line x=y. Hence, we are dealing with a 1-dimensional problem. The original coordinate system, $(x, y)$, is not wrong: it is just not handy as it makes the problem look like 2D. Thus, we change our coordinate system, such that the new $x$-axis coincides with the original x=y line.
3. N2: $m\frac{dv}{dt} = -bv$ with initial conditions: $t = 0 \rightarrow x = 0$ and $t = 0 \rightarrow v = v_0$
4. $\frac{dv}{dt} - \frac{b}{m}v = 0 \rightarrow v = Ae^{-\frac{b}{m}t}$ initial condition: $t = 0 \rightarrow v = v_0 \Rightarrow A = v_0$ Thus: $v(t) = v_0 e^{-\frac{b}{m}t}$
5. for $t \rightarrow \infty : v \rightarrow 0$. The particle comes to rest and then, obviously, the friction force is zero.

**Solution to Exercise 9: Parabolic trajectory with maximum area**

1.



2. We expect that the area, $A$, under the trajectory of the ball is a function of $v$, $g$, and $\theta$. In a dimensional analysis we write this as 'product of powers':

$$A = v^a \cdot g^b \cdot \theta^c \tag{2.140}$$

and we make this expression dimensional correct. (Note: we don't mean that the final outcome of a full analysis is a product of powers, it can be any function but the units should be related in the right way and that is what this 'trick' with powers ensures.)

The area has units m$^2$, velocity m/s, g m/s$^2$ and $\theta$ is dimensionless (radians don't count as a dimension or unit). Thus:

$$\begin{aligned} m : 2 &= a + b \\ s : 0 &= -a - 2b \end{aligned} \tag{2.141}$$

This yields: $a = 4, b = -2$. Thus on dimensional grounds we may expect: $A \sim \frac{v^4}{g^2}$.

3. In the x-direction: no forces, hence $m\frac{v_x}{dt} = 0 \to x(t) = v \cos\theta t$

In the y-direction: $m\frac{dv_y}{dt} = -mg \to y(t) = v \sin\theta t - \frac{1}{2}gt^2$. Where we have used the initial conditions: $x(o) = 0, y(0) = 0, v_x(0) = v\cos\theta, v_y(0) = v \sin\theta$

4. Total time in the air: $v_y(t^*) = 0 \to t^* = \frac{2v}{g}\sin\theta$

5+6. Evaluate the area under the trajectory:

$$\begin{aligned} A &= \int_0^{x_{max}} y dx \\ &= \int_0^{t^*} \left( v \sin\theta t - \frac{1}{2}gt^2 \right) v \cos\theta dt \\ &= v^2 \sin\theta \cos\theta \frac{1}{2}(t^*)^2 - /frac16gv\cos\theta(t^*)^3 \\ &= \frac{2}{3}\frac{v^4}{g^2}\cos\theta\sin^3\theta \end{aligned} \tag{2.142}$$

7. We maximize the function $f(\theta) = \cos\theta \sin^3\theta$:

$$\frac{df}{d\theta} = \sin^2\theta(-\sin^2\theta + 3\cos^2\theta) \tag{2.143}$$

$$\frac{df}{d\theta} = 0 \to \sin\theta = 0 or \sin^2\theta = 3\cos^2\theta \tag{2.144}$$

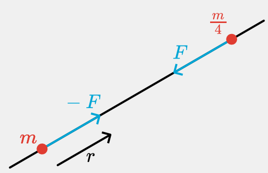
The first solution give a minimum for the area ($A = 0$). So we need the second solution:

$$\frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta = 3 \to \tan\theta = \sqrt{3} \to \theta = \frac{\pi}{3} \tag{2.145}$$

## Solution to Exercise 10: Two attracting particles

### Interpret

We start with a sketch.



This is a 1-dimensional problem. We will use $r$ as the coordinate. Moreover, it is a problem involving two particles, that both can move. This makes it more difficult than 1-dimensional cases with only one particle.

**Develop**

We have to set up two equations of motion, one for particle 1 with mass $m$ and position $r_1$ and one for particle 2 with mass $m/4$ and position $r_2$. When doing so, we should realize that the mutual force obeys Newton's third law: $F_{12} = -F_{21}$

$$\begin{aligned} m\frac{dv_1}{dt} &= a(r_2 - r_1) \\ \frac{m}{4}\frac{dv_2}{dt} &= -a(r_2 - r_1) \end{aligned} \tag{2.146}$$

We see that the two equations are coupled: we can't solve one without information from the other.

**Evaluate**

So, how do we proceed? First, let's think about the question. We are not asked to solve the equation of motion and find the trajectory. What we need to find is the position of the collision.

From the two equation of motion we can find important information about the velocities of both particles. Just add to two equations:

$$m\frac{dv_1}{dt} + \frac{m}{4}\frac{dv_2}{dt} = 0 \rightarrow \frac{dv_1}{dt} = -\frac{1}{4}\frac{dv_2}{dt} \tag{2.147}$$

Since both particles start rest, we find from the last equation: $v_1 = -\frac{1}{4}v_2$ at any time. Thus particle 2 will travel 4 times a distance than particle 1 in the same time interval. Consequently: if particle 1 has moved 1cm, particle 2 has moved 4cm. Thus the particles (originally separated by 5cm) will collide at $r = 1$cm.

**Assess**

It makes sense that the heavy particle has traveled less than the light one: they both feel at any moment the same force (apart from a sign). The light particle will accelerate faster than the heavy one. Moreover, they should collide somewhere on the line element originally separating them as they are attracted to each other.

We found both these elements in our solution.

### 2.2.5.3 Exercises set 2

```
interactive(children=(FloatSlider(value=0.7853981633974483,
description='theta', max=1.5707963267948966, min=0…
```

```
<function __main__.update(theta, F_girl)>
```

```
interactive(children=(FloatSlider(value=0.7853981633974483,
description='theta', max=1.5707963267948966, step=…
```

```
<function __main__.update(theta, mu)>
```

```
interactive(children=(IntSlider(value=1, description='force_num', max=3,
min=1), Output()), _dom_classes=('wid…
```

```
<function __main__.update(force_num)>
```

### Exercise 2.56: Who is strongest? 🌶️

Who is strongest? Two strong boys try to keep a rope straight by each pulling hard at one end. A not so strong third person is pulling in the middle of the rope, but at an angle of 90° to the rope. The two strong boys have the task to keep the deviation of the rope to a small value, set by you.

How does the force and the angle depends on the force exerted by the girl?

Figure 2.57: Picture taken from Show the Physics

```
max=15.0, min=1.5), IntSlider(value=1, d…
```

### Exercise 2.58: Dropping a stone from a church tower 🌶️

You drop a stone from a height of 50m the tower of the church. Calculate the velocity of the stone when it hits the ground (ignore friction). In the video you will see on the left a quick and dirty solution, NOT using IDEA. The right hand side uses IDEA and Newton's $2^{nd}$ law.

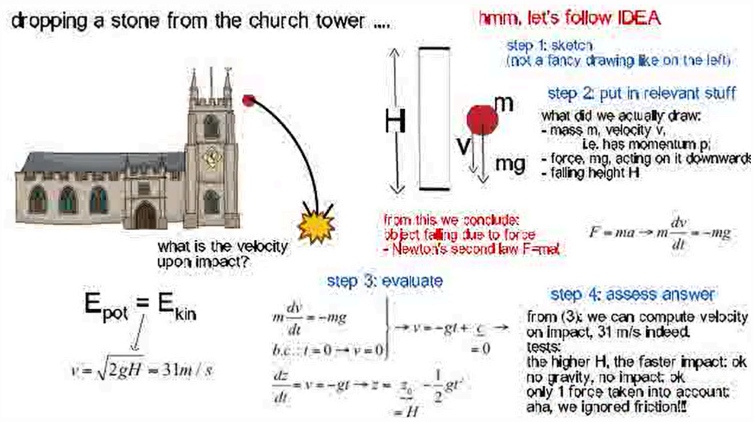



Figure 2.59: The worked out exercise

### Exercise 2.60: Sliding down a slope 🌶️

Two point particles slide down a slope: one feels friction the other doesn't. Can you analyse the situation and understand the graphs?

Exercise 2.61:

Below are three forces and their resultant $(v, t)$- and $(s, t)$-diagrams. What kind of forces are acting?

Exercise 2.62:

A mass $m = 1$kg (the red one in the drawing) is attached to a massless string. The string can move freely over a massless pulley. At the other end of the string a variable mass $M$ (the grey one) is hanging. At $t = 0$ mass $m$ is released, while the string is stretched to its full length.

The graph on the right side of the screen shows the velocity of $m$ as a function of time.

- 'Play' with the acceleration and mass $M$, predict every time first what will happen to the motion.
- Describe the motion of $m$ and $M$.
- Write down Newton's equation of motion for $m$ and for $M$.

```
<function __main__.run_animation(g=9.81, M=1)>
```

Exercise 2.63:

A point particle (mass $m$) is from position $z = 0$ shot with a velocity $v_0$ straight upwards into the air. On this particle only gravity acts, i.e. friction with the air can be ignored. The acceleration of gravity, $g$, may be taken as a constant.

The following questions should be answered.

- What is the maximum height that the particle reaches?
- How long doe it take to reach that highest point?

Solve this exercise using IDEA.

- Sketch the situation and draw the relevant quantities.
- Reason that this exercise can be solved using $\vec{F} = m\vec{a}$ (or $d\vec{p}/dt = \vec{F}$).
- Formulate the equation of motion (N2) for m.
- Classify what kind of mathematical equation this is and provide initial or boundary conditions that are needed to solve the equation.
- Solve the equation of motion and answer the two questions.
- Check your math and the result for dimensional correctness. Inspect the limit: $F_{zw} \to 0$.

### Exercise 2.64: Acceleration of Gravity 🌶️ 🌶️

- Find an object that you can safely drop from some height.
- Drop the object from any (or several heights) and measure using a stop watch or you mobile the time from dropping to hitting the ground.
- Measure the dropping height.

Find from these data the value of gravity's acceleration constant.

Don't forget to first make an analysis of this experiment in terms of a physical model and make clear what your assumptions are.

**Tip**

Think about the effect of air resistance: is dropping from a small, a medium or a high height best? Any arguments?

### Exercise 2.65: Use numerical analysis to assess influence of air friction 🌶️ 🌶️ 🌶️

If you want to learn also how to use numerical methods ...

Try using an air drag force: $F_{drag} = -A_{\perp} C_D \frac{1}{2} \rho_{air} v^2$. With $A_{\perp}$ the cross-sectional area of your object perpendicular to the velocity vector and $C_D \approx 1$ the drag coefficient (in real life it is actually a function of the velocity). $\rho_{air}$ is the density of air which is about $1.2 kg/m^3$.

Write a computer program (e.g. in python) that calculates the motion of your object. See Solution with Python how you could do that.

### Exercise 2.66: Forces on your bike 🌶️

Figure 2.67: Riding a bicycle. Adapted from InjuryMap, from Wikimedia Commons, licensed under CC BY-SA 4.0.

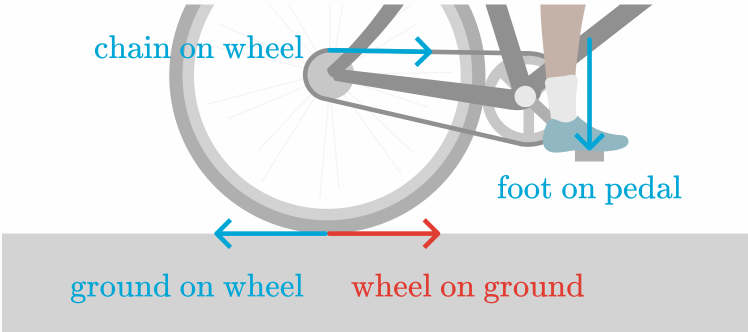
On a bicycle you will have to apply a force to the pedals to move forward, right? What force actually moves you forward, where is it located and who/what is providing that force?

- Make sketch and draw the relevant force. Give the force that actually propels you a different color.
- Think for a minute about the nature of this force: are you surprised?

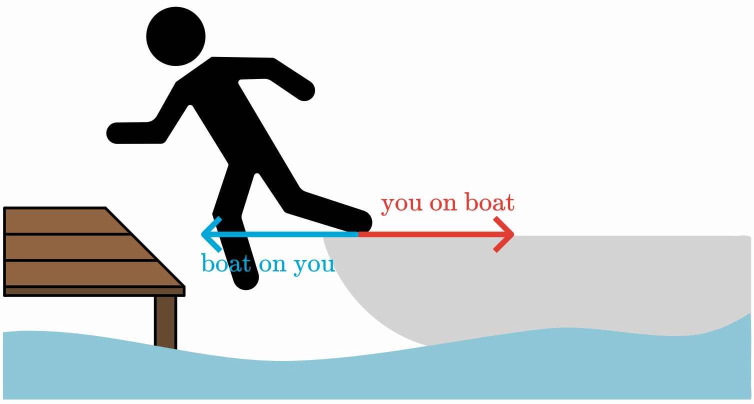
N.B. Consider while thinking about this problem: what would happen if you were biking on an extremely slippery floor?

Solution 2.68: Solution to Exercise 9

When you push with your foot on the pedal, that force is transferred to the chain of your bike. That chain exerts a force on the gear of your bike's rear wheel, trying to get it to rotate. Your wheel touches the ground and, because of the force on the gear, the wheel exerts a force in the ground, trying to push the ground backwards. Due to action=-reaction, the ground exerts a forward force on your wheel. So actually, biking means "making the ground push you forward"!




Exercise 2.70: Getting off the boat 🌶️




You are stepping from a boat onto the shore. Use Newton's laws to describe why you will end up in the water.

N.B. A calculation is not required, but focus on the physics and describe in words why you didn't make it to the jetty.

Solution 2.72: Solution to Exercise 10

When you try to step on the jetty, a force needs to be exerted on you, otherwise you can't move forward. The way you achieve that: you push with your back foot on the boat. And as a result of Newton 3, the boat will push back, but the force from the boat on you is forward directed. That is exactly what you need!

However, while you push, the boat will move backwards due to the force you exert on it. Consequently, your point of contact with the boat shifts away from the jetty. Either you let the boat go and no force from the boat is acting on you. Now gravity will do its work and if your forward velocity is not sufficient, you will not reach the jetty. Or your foot will try to follow the boat and that requires a force to the wrong direction acting on you.

Pushing harder seems an option: your forward velocity might increase more. However, the boat will also be pushed harder and moves quicker away from you. Consequently, the time interval of contact with the boat decreases. Thus, with Newton 2: dp = Fdt your increase in velocity due to the larger force might be compensated by a smaller duration that the force can do so. And you may still end up in the water.

Exercise 2.73: Newton's Laws


Figure 2.74: align: center

Stamp designs © Royal Mail Group Ltd[^1].

Close this book (or don't peak at it ;-)) and write down Newton's laws. Explain in words the meaning of each of the laws. Try to come up with several, different ways of describing what is in these equations.

## 2.3 Work and Energy

Updated: 18 jan 2026

### 2.3.1 Work

Work and energy are two important concepts. Work is the transfer of energy that occurs when a force is applied to an object and causes displacement in the direction of that force, calculated as 'force times path'. However, we need a formal definition:

*if a point particle moves from $\vec{r}$ to $\vec{r} + d\vec{r}$ and during this interval a force $\vec{F}$ acts on the particle, then this force has performed an amount of work equal to:*

$$dW = \vec{F} \cdot d\vec{r} \tag{2.148}$$

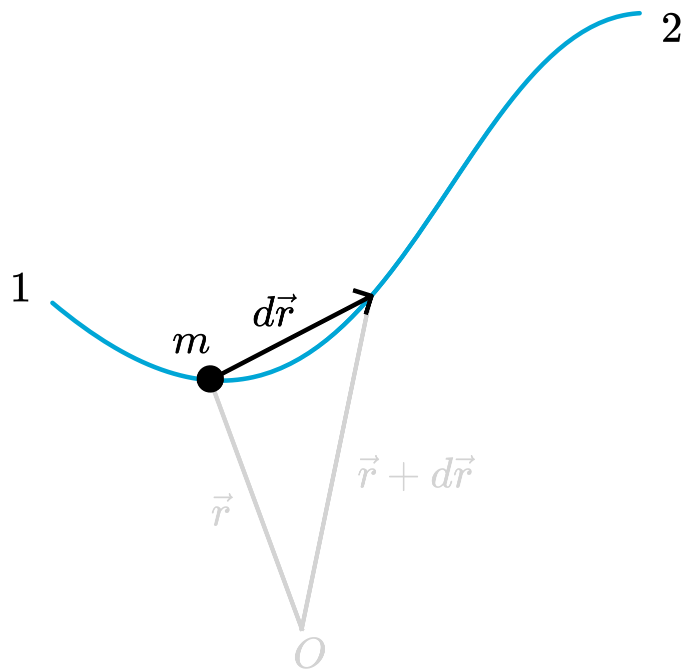



Figure 2.75: Path of a particle.

Note that this an *inner product* between two vectors, resulting in a *scalar* . In other words, work is a number, not a vector. It has no direction. That is one of the advantages over force.

$$dW = \vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz \tag{2.149}$$

Work done on $m$ by $F$ during motion from 1 to 2 over a prescribed trajectory:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} \tag{2.150}$$

Keep in mind: in general the work depends on the starting point 1, the end point 2 and on the trajectory. Different trajectories from 1 to 2 may lead to different amounts of work.

> **Tip**
>
> See also the chapter in the linear algebra book on the inner product

### 2.3.2 Kinetic Energy

Kinetic energy is defined and derived using the definition of work and Newton's $2^{nd}$ Law.

The following holds: if work is done on a particle, then its kinetic energy must change. And vice versa: if the kinetic energy of an object changes, then work must have been done on that particle. The following derivation shows this.

$$
\begin{aligned}
W_{12} &= \int_1^2 \vec{F} \cdot d\vec{r} = \int_1^2 \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_1^2 \vec{F} \cdot \vec{v} dt \\
&= \int_1^2 m \frac{d\vec{v}}{dt} \cdot \vec{v} dt = m \int_1^2 \vec{v} \cdot d\vec{v} = m \left[ \frac{1}{2} \vec{v}^2 \right]_1^2 \\
&= \frac{1}{2} m \vec{v}_2^2 - \frac{1}{2} m \vec{v}_1^2
\end{aligned} \tag{2.151}
$$

It is from the above that we indicate $\frac{1}{2} m \vec{v}^2$ as kinetic energy. It is important to realize that the concept of kinetic energy does not bring anything that is not contained in N2 to the table. But it does give a new perspective: kinetic energy can only be gained or lost if a force performs work on the particle. And vice versa: if a force performs work on a particle, the particle will change its kinetic energy.

Obviously, if more than one force acts, the net work done on the particle determines the change in kinetic energy. It is perfectly possible that force 1 adds an amount $W$ to the particle, whereas at the same time force 2 will take out an amount $-W$. This is the case for a particle that moves under the influence of two forces that cancel each other: $\vec{F}_1 = -\vec{F}_2$. From Newton 2, we immediately infer that if the two forces cancel each other, then the particle will move with a constant velocity. Hence, its kinetic energy stays constant. This is completely in line with the fact that in this case the net work done on the particle is zero:

$$
W_1 + W_2 = \int_1^2 \vec{F}_1 \cdot d\vec{r} + \int_1^2 \vec{F}_2 \cdot d\vec{r} = \int_1^2 \vec{F}_1 \cdot d\vec{r} - \int_1^2 \vec{F}_1 \cdot d\vec{r} = 0 \tag{2.152}
$$

### 2.3.3 Worked Examples

**Exercise 1: Carrying a weight**

You carry a heavy backpack $m = 20$ kg from Delft to Rotterdam (20 km). What is the work that you have done against the gravitational force?

**Solution to Exercise 1: Carrying a weight**

The answer is, of course, zero! That is because the path (from Delft to Rotterdam) is perpendicular to the gravitational force. Therefore the inner product $\vec{F}_g \cdot d\vec{r} = 0$ over the whole way. Let us look at it more formally, this will help us when things get more complicated later.

The force is $\vec{F}_g(x, y, z) = (0, 0, -mg) = -mg\hat{z}$ and we choose our coordinate system such that the path be along the $x$-axis, the $y$-coordinate is zero and we the backpack is at height $z = 1$ m.

$$
W_g = \int_{Delft}^{Rott} F_x dx + F_y dy + F_z dz = \int F_x dx \,|_{y=0,z=1} = \int 0 \, dx = 0 \tag{2.153}
$$

So gravity has not performed work on your backpack. Similarly, you have exercised a force $\vec{F}_N$ on the backpack. As the backpack doesn't change its vertical coordinate, we know $\vec{F}_N + \vec{F}_g = 0$. And immediately, we see:

$$
W_N = \int_{Delft}^{Rott} F_{Nx} dx + F_{Ny} dy + F_{Nz} dz = \int F_x dx \,|_{y=0,z=1} = \int 0 \, dx = 0 \tag{2.154}
$$

You didn't perform any work either. This may feel strange or even wrong. After all, you will probably be pretty tired after the walk. However, that is due to the internal working of our muscles and body. In order to sustain the force $\vec{F}_N$ humans do use energy: work is done in their muscles. But from a physics point of view: no work is done on the backpack.

**Exercise 2: Compressing a spring**[5]

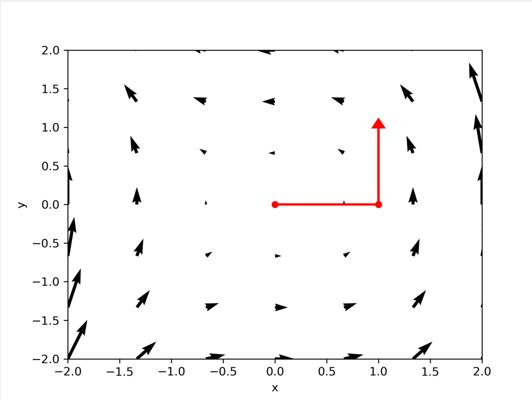
You're compressing an uncompressed spring with spring constant $k$ over a distance $x$. How much work do you need to do?

**Solution to Exercise 2: Compressing a spring**

$$W = \int_{x_1}^{x_2} F \mathrm{d}x = \int_0^x kx \mathrm{d}x = \frac{1}{2}kx^2 \tag{2.155}$$

**Exercise 3: Work in a force field**

Now we consider a force field $\vec{F}(x,y) = (-y, x^2) = -y\hat{x} + x^2\hat{y}$. We compute the work done over a path from the origin $(0,0)$ to $(1,0)$ and then to $(1,1)$ first along the $x$-axis and then parallel to the $y$-axis.




**Solution to Exercise 3: Work in a force field**

We can split up the integral in these two parts as the direction in both parts is constant, therefore the inner product can be separated out.

$$\begin{aligned} W &= \int_{(0,0)}^{(1,0)} \vec{F} \cdot d\vec{r} + \int_{(1,0)}^{(1,1)} \vec{F} \cdot d\vec{r} \\ &= \int_{(0,0)}^{(1,0)} F_x dx \mid_{y=0} + \int_{(1,0)}^{(1,1)} F_y dy \mid_{x=1} \\ &= \int_{(x=0)}^{(x=1)} -y \, dx \mid_{y=0} + \int_{(y=0)}^{(y=1)} x^2 \, dy \mid_{x=1} \\ &= -yx \mid_{x=0}^{x=1} \mid_{y=0} + x^2 y \mid_{y=0}^{y=1} \mid_{x=1} = 1 \end{aligned} \tag{2.156}$$

Try to integrate the force field yourself along a different path $(0,0) \to (0,1) \to (1,1)$ to the same end point.

$$\begin{aligned} W &= \int_{y=0}^{y=1} F_y \, dy \mid_{x=0} + \int_{x=0}^{x=1} F_x \, dx \mid_{y=1} \\ &= \int_{y=0}^{y=1} x^2 \, dy \mid_{x=0} + \int_{x=0}^{x=1} -y \, dx \mid_{y=1} \\ &= -1 + 0 = -1 \end{aligned} \tag{2.157}$$

The work done is not the same over this path. This is already obvious from the graph showing the path and the force field: the second path clearly moves against the force, where the first is moving with direction of the force.

**Reminder of path/line integral from Analysis**

[5]Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

As long as the path can be split along coordinate axis the separation above is a good recipe. If that is not the case, then we need to turn back to the way how things have been introduced in the Analysis class. We need to make a 1D parameterization of the path.

Line integral of a vector valued function $\vec{F}(x, y, z) : \mathbb{R}^3 \to \mathbb{R}^3$ over a curve `cal` $C$ is given as

$$\int_{\texttt{cal}\ C} \vec{F}(x, y, z) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(\tau)) \cdot \frac{d\vec{r}(\tau)}{d\tau}\, d\tau \tag{2.158}$$

We integrate in the definition of the work from point 1 to 2 over an implicitly given path. To compute this actually, you need to parameterize the path by $\vec{r}(\tau) = (x(\tau), y(\tau), z(\tau))$. The integration variable $\tau$ tells you where you are on the path, $\tau \in [a, b] \in \mathbb{R}$. The derivative of $\vec{r}$ with respect to $\tau$ gives the tangent vector to the curve, the "speed" of walking along the curve. In the analysis class you used $\vec{v}(\tau) \equiv \frac{d\vec{r}(\tau)}{d\tau}$ for the speed. The value of the line integral is independent of the chosen parameterization. However, it changes sign when reversing the integration boundaries.

**Example: Another path**

Now we integrate from $(0, 0) \to (1, 1)$ but along the diagonal. A parameterization of this path is $\vec{r}(\tau) = (0, 0) + (1, 1)\tau = (\tau, \tau), \tau \in [0, 1]$. The derivative is $\frac{d\vec{r}(\tau)}{d\tau} = (1, 1)$. Therefore we can write the work of $\vec{F}(x, y) = -y\hat{x} + x^2\hat{y}$ along the diagonal as

$$\begin{gathered}\int_0^1 \vec{F}(\tau, \tau) \cdot (1, 1)\, d\tau = \int_0^1 (-\tau, \tau^2) \cdot (1, 1)\, d\tau = \\ \int_0^1 (-\tau + \tau^2)\, d\tau = -\frac{1}{6}\end{gathered} \tag{2.159}$$

Integration of the same force $\vec{F}(x, y) = -y\hat{x} + x^2\hat{y}$ from $(0, 0) \to (1, 1)$ but along a normal parabola. A parameterization of the path is $\vec{r}(\tau) = (0, 0) + (\tau, \tau^2), \tau \in [0, 1]$ and the derivative is $\frac{d\vec{r}}{d\tau} = (1, 2\tau)$. The work then is

$$\begin{gathered}\int_0^1 \vec{F}(\tau, \tau^2) \cdot (1, 2\tau)\, d\tau = \\ \int_0^1 (-\tau^2, \tau^2) \cdot (1, 2\tau)\, d\tau = \\ \int_0^1 (-\tau^2 + 2\tau^3)\, d\tau = \frac{1}{6}\end{gathered} \tag{2.160}$$

### 2.3.4 Gravitational potential energy

Let's consider an object close to the surface of any planet, where the acceleration due to gravity can be described by $F_g = -mg$. Raising the object to a height $H$ requires us to do work: we will have to apply a force $F = +mg$ to the object to lift it to position $H$. Thus, with two forces acting - each doing work on the object we get:

$$\begin{aligned}W_g &= \int_0^H F_g dx = \int_0^H -mg dx = -mgH \\ W_+ &= \int_0^H -F_g dx = \int_0^H mg dx = mgH\end{aligned} \tag{2.161}$$

The net effect is of course $W_{net} = 0$ as the object started without kinetic energy and ends without kinetic energy, thus we knew in advance $0 = \Delta E_{kin} = W_g + W_+$

We can also take a slightly different view on this. Suppose we only concentrate on the work done by gravity: $W_g = -mgH$. Note that there is a minus sign, the gravitational force works in the opposite direction of the movement of the object. As energy is a conservative quantity, someone or something has supplied the object with some 'gained' energy. We call this potential energy, more particular in this case gravitational potential energy.

Why is it called 'potential'? When the object is released from that height $H$, this gravitational potential energy is converted to kinetic energy. The gravitational force does work on the object:

$$W = \int_H^0 F dx = \int_H^0 mgdx = mgH = \Delta E_{kin} \tag{2.162}$$

From this, it follow that the object will reach a velocity of $v = \sqrt{2gH}$. This is an example of a situation where an object looses potential energy and gains kinetic energy.

**Exercise 4: Potential & kinetic energy**

Proof that the velocity of an object released from a height $H$ will reach the velocity $v = \sqrt{2gH}$.

**Note**

Exercise 5: A point particle of mass $m = 1$kg is at $t = 0$ at position $x = 0$. It has initial velocity $v_0$. From $t = 0$ to $t_{stop} = 2$s it is under the influence of a constant force $F$. This is a 1D problem.

The top graph show the position of the particle. The bottom graph shows the Work done on the particle by the force and the kinetic energy of the particle.

Analyse this situation and calculate the work done by the force at any time. Is the work done in this case always sufficient to account for the change in kinetic energy? What does it mean if the work is positive or negative?

**Note**

Exercise 6: Use the Python app below, and answer the following questions:

- does the acceleration double when the mass of the falling box doubles?
- the position time diagram is made using kinematics, how would the code look like when based on energy conservation?
- how would you include friction in the code?

```
interactive(children=(FloatSlider(value=9.81, description='g (m/s²)',
max=15.0, min=1.5), IntSlider(value=1, d…
```

```
<function __main__.run_animation(g=9.81, M=1)>
```

Exercise 2.77:

Look at the following roller coaster app.

Change the various graph settings (what is on the x/y axis). Change the starting position of the ball, and try to change the path.

Can you make sense of the motion and the graphs?

### 2.3.5 Conservative force

As we saw, work done on $m$ by $F$ during motion from 1 to 2 over a prescribed trajectory, is defined as:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} \tag{2.163}$$

In general, the amount of work depends on the path followed. That is, the work done when going from $\vec{r}_1$ to $\vec{r}_2$ over the red path in the figure below, will be different when going from $\vec{r}_1$ to $\vec{r}_2$ over the blue path. Work depends on the specific trajectory followed.

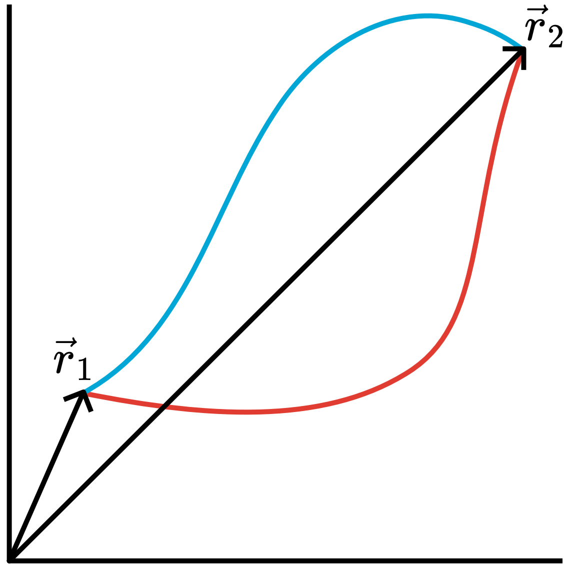



Figure 2.78: Two different paths.

However, there is a certain class of forces for which the path does not matter, only the start and end point do. These forces are called conservative forces. As a consequence, the work done by a conservative force over a closed path, i.e start and end are the same, is always zero. No matter which closed path is taken.

$$conservative force \Leftrightarrow \oint \vec{F} \cdot d\vec{r} = 0 for ALL closed paths \tag{2.164}$$

#### 2.3.5.1 Stokes' Theorem

It was George Stokes who proved an important theorem, that we will use to turn the concept of conservative forces into a new and important concept.

Figure 2.79: Sir George Stokes (1819-1903). From Wikimedia Commons, public domain.

His theorem reads as:

$$\oint \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} \tag{2.165}$$

In words: the integral of the force over a closed path equals the surface integral of the curl of that force. The surface being 'cut out' by the close path. The term $\vec{\nabla} \times \vec{F}$ is called the curl of $F$:, which is a vector. The meaning of the curl and some words on the theorem are given below.

**Intermezzo: intuitive proof of Stokes' Theorem**

Consider a closed curve $\Gamma$ in the $xy$-plane. We would like to calculate the work done when going around this curve. In other words: what is $\oint \vec{F} \cdot d\vec{r}$ if we move along this curve?

We can visualize what we need to do: we cut the curve in small part; compute $\vec{F} \cdot d\vec{r}$ for each part (i.e. the red, green, blue, etc. in Figure 5 and sum these to get the total along the curve. If we make the parts infinitesimally small, we go from a (Riemann) sum to an integral.

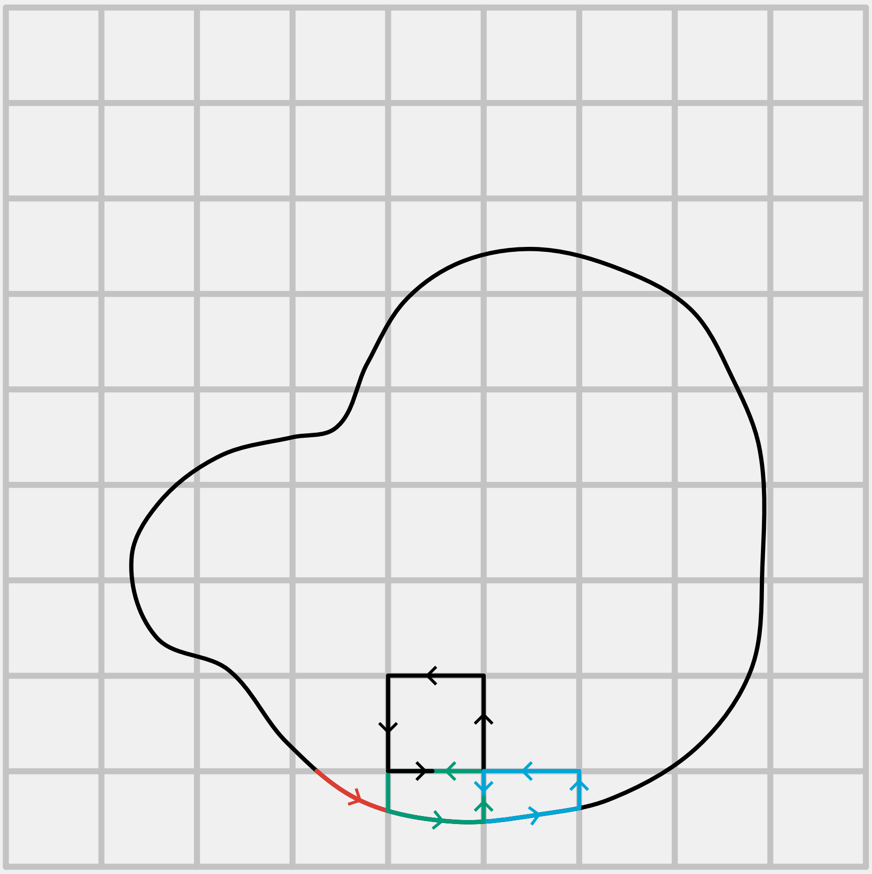


Figure 2.80: Closed path on a grid.

However, we are going to compute much more: take a look at Figure 5. We have put a grid in the $xy$-plane over a closed curve $\Gamma$. Hence, the interior of our curve is full of squares. We are not only computing the parts along the curve, but also along the sides of alle curves. This will sound like way too much work, but we will see that it actually is a very good idea.

See Figure 5: we calculate $\oint \vec{F} \cdot d\vec{r}$ counter clockwise for the green square. Than we have at least the green part of our $\oint \vec{F} \cdot d\vec{r}$ done in the right direction. Hence, we compute $\int \vec{F} \cdot d\vec{r}$ along the right side of the green square. We do that from bottom to top as we go counter clockwise along the green square. Let's call that $\int_g \vec{F} \cdot d\vec{r}$.

Then we move to the blue square and repeat in counter clockwise direction our calculation. But this means that we compute along the left side of blue the square from top to bottom. We will call this $\int_b \vec{F} \cdot d\vec{r}$.

Next, we will add all contributions. Thus we get $\int_g \vec{F} \cdot d\vec{r} + \int_b \vec{F} \cdot d\vec{r}$. But these two cancel each other as they are exactly the same but done in opposite directions. Thus if we use that $\int_1^2 f dx = -\int_2^1 f dx$ for any integration, it becomes obvious that $\int_g \vec{F} \cdot d\vec{r} + \int_b \vec{F} \cdot d\vec{r} = 0$.

Note that this will happen for all side of the squares that are in the interior of our curve. Thus, the integral over all squares is exactly the integral along the curve $\Gamma$.

It seems, we do a lot of work for nothing. But there is another way of looking at the path-integrals along the squares. If we make the square small enough, the calculation along one square can be approximated:

$$\begin{aligned}\oint_{square} \vec{F}\cdot d\vec{r} &\approx F_x(x,y)dx + F_y(x+dx,y)dy - F_x(x,y+dy)dx - F_y(x,y)dy \\ &\approx \frac{F_x(x,y)-F_x(x,y+dy)}{dy}dxdy + \frac{F_y(x+dx,y)-F_y(x,y)}{dx}dxdy \\ &\approx \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)dxdy\end{aligned} \tag{2.166}$$

The results get more accurate the smaller we make the square.

If we now sum up all squares and make these squares infinitesimally small, the sum becomes an integral, but now an integral over the surface enclosed by the curve:

$$\oint_\Gamma \vec{F}\cdot d\vec{r} = \iint \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)dxdy \tag{2.167}$$

The right hand side of the above equation is a surface integral of the 'vector' $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$. Obviously, we did not provide a rigorous proof, but only an intuitive one. For a mathematical proof, see your calculus classes.

Moreover, we only worked in the $xy$-plane. If we would extend our reasoning to a closed curve in 3 dimensions, we would get Stokes theorem, which reads as:

$$\oint_\Gamma \vec{F}\cdot d\vec{r} = \iint \vec{\nabla}\times\vec{F}\cdot d\vec{\sigma} \tag{2.168}$$

Here, $d\vec{\sigma}$ is a small element out of the surface. Note that it is a vector: it has size and directions. The latter is perpendicular to the surface element itself. Furthermore, we have the vector $\vec{\nabla}\times\vec{F}$. This is the cross-product of the nabla-operator and our vector field $\vec{F}$. The nabla operator is a strange kind of vector. Its components are: partial differentiation. In a Cartesian coordinate system this can be written as:

$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \tag{2.169}$$

or if you prefer a column notation:

$$\vec{\nabla} \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z}\end{pmatrix} \tag{2.170}$$

The curl of $F$ can be found from e.g.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z} \tag{2.171}$$

Note of warning: do be careful with the nabla-operator. It is not a standard vector. For instance, ordinary vectors have the property $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$. This does not hold for the nabla-operator.

Second note of warning: the representation of the nabla-operator does change quite a bit when using other coordinate systems like cylindrical or spherical. For instance, in cylindrical coordinates it is ***not*** equal to $\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z}\end{pmatrix}$. This can be easily seen as both $r, z$ have units length, i.e. meters, but $\phi$ has no units.

**Example: Work done in a vectorfield**

Suppose we need to calculate the integral of the vectorfield $\vec{F}(x, y) = y\hat{x} - x\hat{y}$ over the closed curve formed by a square from $(0, 0)$ to $(1, 0)$, $(1, 1)$, $(0, 1)$ and back to $(0, 0)$.

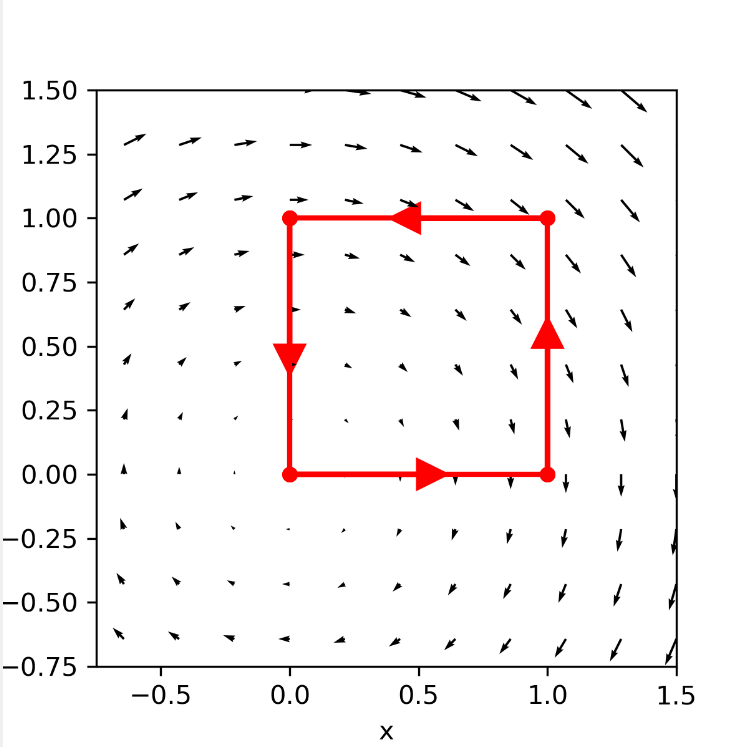



Figure 2.81: Integrating along the unit square.

We go counter clockwise.

$$\begin{aligned}
\oint \vec{F} \cdot d\vec{r} &= \int_{x=0}^{1} F_x(x, y = 0)dx + \int_{y=0}^{1} F_y(x = 1, y)dy + \\
&+ \int_{x=1}^{0} F_x(x, y = 1)dx + \int_{y=1}^{0} F_y(x = 0, y)dy \\
&= \int_0^{10} dx + \int_0^{1} -1dy + \int_1^{01} dx + \int_1^{0} -0dx \\
&= 0 - [y]_0^1 + [x]_1^0 - 0 \\
&= -2
\end{aligned} \tag{2.172}$$

Now we try this using Stokes' Theorem:

$$\oint \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} \tag{2.173}$$

We first calculate $\vec{\nabla} \times \vec{F}$:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \left( \frac{\partial(-x)}{\partial x} - \frac{\partial(y)}{\partial y} \right) \hat{z} = -2\hat{z} \tag{2.174}$$

Thus, in this example $\vec{\nabla} \times \vec{F}$ has only a $z$-component.

An elementary surface element of the square is: $d\vec{\sigma} = dxdy\hat{z}$. This also has only a $z$-component. Note that it points in the positive $z$-direction. This is a consequence of the counter clockwise direction that we use to go along the square.

According to Stokes Theorem, we this find:

$$\oint \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} = \int_{x=0}^{1} \int_{y=0}^{1} (-2) dx dy = -2 \tag{2.175}$$

Indeed, we find the same outcome.

#### 2.3.5.2 Conservative force and $\vec{\nabla} \times \vec{F}$

For a conservative force the integral over the closed path is zero for any closed path. Consequently, $\vec{\nabla} \times \vec{F} = 0$ everywhere. How do we know this? Suppose $\vec{\nabla} \times \vec{F} \neq 0$ at some point in space. Then, since we deal with continuous differentiable vector fields, in the close vicinity of this point, it must also be non-zero. Without loss of generality, we can assume that in that region $\vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} > 0$. Next, we draw a closed curve around this point, in this region. We now calculate the $\oint \vec{F} \cdot d\vec{r}$ along this curve. That is, we invoke Stokes Theorem. But we know that $\vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} > 0$ on the surface formed by the closed curve. Consequently, the outcome of the surface integral is non-zero. But that is a contradiction as we started with a conservative force and thus the integral should have been zero.
The only way out, is that $\vec{\nabla} \times \vec{F} = 0$ everywhere.

Thus we have:

$$conservative force \Leftrightarrow \vec{\nabla} \times \vec{F} = 0 everywhere \tag{2.176}$$

### 2.3.6 Potential Energy

This function $V$ is called the potential energy or the potential for short and has a direct connection to the work. A direct consequence of the above is:

if $\vec{\nabla} \times \vec{F} = 0$ everywhere, a function $V(\vec{r})$ exists such that $\vec{F} = -\vec{\nabla} V$

$$\begin{gathered} conservative force \Leftrightarrow \vec{\nabla} \times \vec{F} = 0 everywhere \\ \Updownarrow \\ \vec{F} = -\vec{\nabla} V \Leftrightarrow V(\vec{r}) = -\int_{ref} \vec{F} \cdot d\vec{r} \end{gathered} \tag{2.177}$$

where in the last integral, the lower limit is taken from some, self picked, reference point. The upper limit is the position $\vec{r}$.

Next to its direct connection to work, the potential is also connected to kinetic energy.

$$E_{kin,2} - E_{kin,1} = W_{12} = \int_{1}^{2} \vec{F} \cdot d\vec{r} = V(\vec{r}_2) - V(\vec{r}_1) \tag{2.178}$$

or rewritten:

$$E_{kin,1} + V(\vec{r}_1) = E_{kin,2} + V(\vec{r}_2) \tag{2.179}$$

In words: for a conservative force, the sum of kinetic and potential energy stays constant.

#### 2.3.6.1 Energy versus Newton's Second Law

We, starting from Newton's Laws, arrived at an energy formulation for physical problems.
Question: can we also go back? That is: suppose we would start with formulating the energy rule for a physical problem, can we then back out the equation of motion?
Answer: yes, we can!

It goes as follows. Take a system that can be completely described by its kinetic plus potential energy. Then: take the time-derivative and simplify, we will do it for a 1-dimensional case first.

$$\begin{aligned}
&\frac{1}{2}mv^2 + V(x) = E_0 \Rightarrow \\
&\frac{d}{dt}\left[\frac{1}{2}mv^2 + V(x)\right] = \frac{dE_0}{dt} = 0 \Rightarrow \\
&mv\dot{v} + \frac{dV}{dx}\underbrace{\frac{dx}{dt}}_{=v} = 0 \Rightarrow \\
&v\left(m\dot{v} + \frac{dV}{dx}\right) = 0
\end{aligned} \tag{2.180}$$

The last equation must hold for all times and all circumstances. Thus, the term in brackets must be zero.

$$m\dot{v} + \frac{dV}{dx} = 0 \Rightarrow m\ddot{x} = -\frac{dV}{dx} = F \tag{2.181}$$

And we have recovered Newton's second law.

In 3 dimensions it is the same procedure. What is a bit more complicated, is using the chain rule. In the above 1-d case we used $\frac{dV}{dt} = \frac{dV(x(t))}{dt} = \frac{dV}{dx}\frac{dx(t)}{dt}$. In 3-d this becomes:

$$\frac{dV}{dt} = \frac{dV(\vec{r}(t))}{dt} = \frac{dV}{d\vec{r}} \cdot \frac{d\vec{r}(t)}{dt} = \vec{\nabla}V \cdot \vec{v} \tag{2.182}$$

Thus, if we repeat the derivation, we find:

$$\begin{aligned}
&\frac{1}{2}mv^2 + V(\vec{r}) = E_0 \Rightarrow \\
&\frac{d}{dt}\left[\frac{1}{2}mv^2 + V(\vec{r})\right] = 0 \Rightarrow \\
&m\vec{v} \cdot \dot{\vec{v}} + \vec{\nabla}V \cdot \vec{v} = 0 \Rightarrow \\
&v\left(m\vec{a} + \vec{\nabla}V\right) = 0 \Rightarrow \\
&m\vec{a} = -\vec{\nabla}V = \vec{F}
\end{aligned} \tag{2.183}$$

And we have recovered the 3-dimensional form of Newton's second Law. This is a great result. It allows us to pick what we like: formulate a problem in terms of forces and momentum, i.e. Newton's second law, or reason from energy considerations. It doesn't matter: they are equivalent. It is a matter of taste, a matter of what do you see first, understand best, find easiest to start with. Up to you!

### 2.3.7 Stable and Unstable Equilibrium

A particle (or system) is in equilibrium when the sum of forces acting on it is zero. Then, it will keep the same velocity, and we can easily find an inertial system in which the particle is at rest, at an equilibrium position.
The equilibrium position (or more general: state) can also be found directly from the potential energy.

Potential energy and (conservative) forces are coupled via:

$$\vec{F} = -\vec{\nabla}V \tag{2.184}$$

The equilibrium positions $\left(\sum_i \vec{F}_i = 0\right)$ can be found by finding the extremes of the potential energy:

$$equilibriumposition \Leftrightarrow \vec{\nabla}V = 0 \tag{2.185}$$

Once we find the equilibrium points, we can also quickly address their nature: is it a stable or unstable solution? That follows directly from inspecting the characteristics of the potential energy around the equilibrium points.

For a stable equilibrium, we require that a small push or a slight displacement will result in a force pushing back such that the equilibrium position is restored (apart from the inertia of the object that might cause an overshoot or oscillation).

However, an unstable equilibrium is one for which the slightest push or displacement will result in motion away from the equilibrium position.

The second derivative of the potential can be investigated to find the type of extremum. For 1D functions that is easy, for scalar valued functions of more variables that is a bit more complicated. Here we only look at the 1D case $V(x) : \mathbb{R} \to \mathbb{R}$

$$equilibrium : \vec{\nabla} V = 0 \tag{2.186}$$

Luckily, the definition of potential energy is such that these rules are easy to visualize in 1D and to remember, see Figure 7

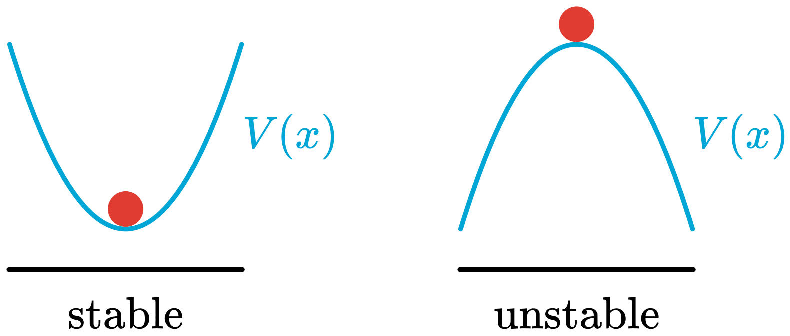



Figure 2.82: Stable and unstable position of a particle in a potential.

A valley is stable; a hill top is unstable.
NB: Now the choice of the minus sign in the definition of the potential is clear. Otherwise a hill would be stable, but that does not feel natural at all.

It is also easy to visualize what will happen if we distort that particle from the equilibrium state:

- The valley, i.e., the stable system, will make the particle move back to the lowest point. Due to inertia, it will not stop but will continue to move. As the lowest position is one of zero force, the particle will 'climb' toward the other end of the valley and start an oscillatory motion.
- The top, i.e., the unstable point, will make the particle move away from the stable point. The force acting on the particle is now pushing it outwards, 'down the slope of the hill'.

#### 2.3.7.1 Taylor Series Expansion of the Potential

The Taylor expansion or Taylor series is a different way of writing down the value of a function in the vicinity of a point $x_0$. Even though the function is written down in a different way, it is equal to $f$ in the vicinity of $x_0$. It uses an infinite series of polynomial terms with coefficients given by value of the derivative of the function at that specific point $x_0$. The value of the terms for higher n become small, so we can approximate the function by using only the first few terms. The more of these first terms you take, the closer your approximation is. Mathematically, it reads for a 1D scalar function $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) \approx f(x_0) + \frac{1}{1!} f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \tag{2.187}$$

For our purpose here, it suffices to stop after the second derivative term:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \mathcal{O}(x^3) \tag{2.188}$$

A way of understanding why the Taylor series actually works is the following.
Imagine you have to explain to someone how a function looks around some point $x_0$, but

you are not allowed to draw it. One way of passing on information about $f(x)$ is to start by giving the value of $f(x)$ at the point $x_0$:

$$f(x) \approx f(x_0) \tag{2.189}$$

Next, you give how the tangent at $x_0$ is: you pass on the first derivative at $x_0$. The other person can now see a bit better how the function changes when moving away from $x_0$:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \tag{2.190}$$

Then, you tell that the function is not a straight line but curved, and you give the second derivative. So now the other one can see how it deviates from a straight line:

$$f(x) \approx f(x_0) + \frac{1}{1!}f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 \tag{2.191}$$

Note that the prefactor is placed back. But the function is not necessarily a parabola; it will start deviating more and more as we move away from $x_0$. Hence we need to correct that by invoking the third derivative that tells us how fast this deviation is. And this process can continue on and on.

Important to note: if we stay close enough to $x_0$ the terms with the lowest order terms will always prevail as higher powers of $(x - x_0)$ tend to zero faster than a lower powers (for instance: $0.5^4 << 0.5^2$).

This 3Blue1Brown clip explains the 1D Taylor series nicely.

Figure 2.83: A 3blue1brown clip on Taylor series.

For scalar valued functions as our potentials $V(\vec{r}) : \mathbb{R}^3 \rightarrow \mathbb{R}$ the extension of the Taylor series is not too difficult. If we expand the function around a point

$$\begin{aligned} V(\vec{r}) &\approx V(\vec{r}_0) + \vec{\nabla}V(\vec{r}_0) \cdot (\vec{r} - \vec{r}_0) \\ &+ \frac{1}{2}(\vec{r} - \vec{r}_0) \cdot (\partial^2 V)(\vec{r}_0) \cdot (\vec{r} - \vec{r}_0) + \mathcal{O}(r^3) \end{aligned} \tag{2.192}$$

The second derivative of the potential indicated by $\partial^2 V$ is the Hessian matrix. Right now, this all sound a bit hocus pocus. But don't worry: you won't need it right away in its full glory. In the rest of your physics and math classes, this will all come back and start to make sense.

Conceptually the extrema of the function are again the hills and valleys. The classification of the extrema has next to hills and valleys also saddle points etc. In this course we will not bother about these more dimensional cases, but only stick to simple ones.

Exercise 2.84: Gravity, a conservative force? 🌶️

Is gravity $\vec{F}_g = m\vec{g}$ a conservative force? If yes, what is the corresponding potential energy?

To find the answer:

a. Show $\vec{\nabla} \times m\vec{g} = 0$
b. Find a $V$ that satisfies $-m\vec{g} = -\vec{\nabla}V$

Exercise 2.85: 🌶️

A point particle of mass $m = 1$ kg is at $t = 0$ at position $x = 0$. It has initial velocity $v_0$. From $t = 0$ to $t_{stop} = 2$ s it is under the influence of a constant force $F$. This is a 1D problem.

The top graph shows the position of the particle. The bottom graph shows the Work done on the particle by the force and the kinetic energy of the particle.

Analyze this situation and calculate the work done by the force at any time. Is the work done in this case always sufficient to account for the change in kinetic energy? What does it mean if the work is positive or negative?

### 2.3.8 Examples, exercises and solutions

Updated: 13 okt 2025

Exercise 2.86: 🌶️

A simple model for the frictional force experienced by a body sliding over a horizontal, smooth surface is $F_f = -\mu F_g$ with $F_g$ the gravitational force on the object. The friction force is opposite the direction of motion of the object.

- Show that this frictional force is not conservative (and, consequently, a potential energy associated does not exist!).

**Tip**

Think of two different trajectories to go from point 1 to point 2 and show that the amount of work along these trajectories is not the same.

Or: find a closed loop for which the work done by the frictional force is non-zero.

Exercise 2.87: 🌶️

A force is given by: $\vec{F} = x\hat{x} + y\hat{y} + z\hat{z}$

- Show that this force is conservative.
- Find the corresponding potential energy.

A second force is given by: $\vec{F} = y\hat{x} + x\hat{y} + z\hat{z}$

- Show that this force is also conservative.
- Find the corresponding potential energy.

Exercise 2.88: 🌶️

Another force is given by: $\vec{F} = y\hat{x} - x\hat{y}$

- Show that this force is not conservative.
- Compute the work done when moving an object over the unit circle in the xy-plane in an anti-clockwise direction. (Hint: use Stokes theorem.)
- Discuss the meaning of your answer: is it positive or negative? And what does that mean in terms of physics?

Exercise 2.89: 🌶️

Given a potential energy $E_{pot} = xy$.
a. Find the corresponding force (field).
b. Make a plot of $\vec{F}$ as a function of (x,y,z).
c. Describe the force and comment on what the potential itself already reveals about the force.

Exercise 2.90: 🌶️

Given a force field $\vec{F} = -xy\hat{x} + xy\hat{y}$. A particle moves from $(x, y) = (0, 0)$ over the x-axis to $(x, y) = (1, 0)$ and then parallel to the y-axis to $(x, y) = (1, 1)$. In a second motion, the same particle goes from $(x, y) = (0, 0)$ over the y-axis to $(x, y) = (0, 1)$ and then parallel to the x-axis to end also in $(x, y) = (1, 1)$.

- Show that not necessarily the work done over the two paths is equal.
- Compute the amount of work done over each of the paths.

Exercise 2.91: 🌶️ 🌶️

A particle of mass m is initially at position $x = 0$. It has zero velocity. On the particle a force is acting. The force can be described by $F = F_0 \sin \frac{x}{L}$ with $F_0$ and $L$ positive constants.

1. Show that this force is conservative and find the corresponding potential. Take as reference point for the potential energy $x = \frac{\pi}{2}L$.
2. The particle gets a tiny push, such that it starts moving in the positive x-direction. Its initial velocity is so small that, for all practical calculations, it can be set to zero. What will happen to the particle after the push?
3. Find the maximum velocity that the particle can get. At which location(s) will this take place?

Note: this is a 1-dimensional problem.

Solution 2.92: Solution to Exercise 1

a. Show $\vec{\nabla} \times m\vec{g} = 0$
$\vec{\nabla} \times m\vec{g} = 0$? How to compute it? For **Cartesian** coordinates there is an easy to remember rule:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{2.193}$$

If we chose our coordinates such that $\vec{g} = -g\hat{z}$ we get:

$$\vec{\nabla} \times \vec{F}_g = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -mg \end{vmatrix} = 0 \tag{2.194}$$

Thus $\vec{F}_g$ is conservative.

b. Find a $V$ that satisfies $-m\vec{g} = -\vec{\nabla}V$
Does $-m\vec{g} = -\vec{\nabla}V$ have a solution for V? Let's try, using the same coordinates as above.

$$\begin{aligned} &-\vec{\nabla}V = -m\vec{g} \Rightarrow \\ &\frac{\partial V}{\partial x} = 0 \rightarrow V(x,y,z) = f(y,z) \\ &\frac{\partial V}{\partial y} = 0 \rightarrow V(x,y,z) = g(x,z) \\ &\frac{\partial V}{\partial z} = mg \rightarrow V(x,y,z) = mgz + h(x,y) \end{aligned} \tag{2.195}$$

f,g,h are unknown functions. But all we need to do, is find one $V$ that satisfies $-m\vec{g} = -\vec{\nabla}V$.

So, if we take $V(x,y,z) = mgz$ we have shown, that gravity in this form is conservative and that we can take $V(x,y,z) = mgz$ for its corresponding potential energy.

By the way: from the first part (curl F = 0), we know that the force is conservative and we know that we could try to find V from

$$\begin{aligned} V(x,y,z) &= -\int_{ref} m\vec{g} \cdot d\vec{r} = \int_{ref} mg\hat{z} \cdot d\vec{r} \\ &= \int_{ref} mgdz = mgz + const \end{aligned} \tag{2.196}$$

Solution 2.93: Solution to Exercise 3

Click for the solution Friction Not Conservative.

Solution 2.94: Solution to Exercise 4

Click for the solution Conservative Force.

Solution 2.95: Solution to Exercise 5

Click for the solution Non-Conservative Force.

Solution 2.96: Solution to Exercise 6

Click for the solution Potential energy & Force.

Solution 2.97: Solution to Exercise 7

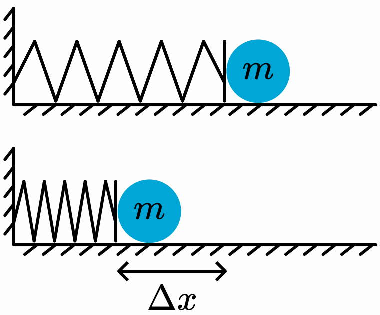
Click for the solution Force Field.

Solution 2.98: Solution to Exercise 8

Click for the solution Sinusoidal Force Field.

Exercise 2.99: Shooting a ball using a spring 🌶️

A ball with mass $m$ is horizontally pressed against a spring with spring constant $k$, compressing the spring by $\Delta x$.




1. Express the velocity of the ball when released.
2. Why is in real life the actual velocity of the ball less (friction is not the answer)?
3. Why is the velocity of the ball less when shot vertically?

### Exercise 2.101: Firing a cannon ball[6] 🌶️

1. Show that, if you ignore drag, a projectile fired at an initial velocity $v_0$ and angle $\theta$ has a range $R$ given by

$$R = \frac{v_0^2 \sin 2\theta}{g} \tag{2.197}$$

1. A target is situated 1.5 km away from a cannon across a flat field. Will the target be hit if the firing angle is 42° and the cannonball is fired at an initial velocity of 121 m/s? (Cannonballs, as you know, do not bounce).
2. To increase the cannon's range, you put it on a tower of height $h_0$. Find the maximum range in this case, as a function of the firing angle and velocity, assuming the land around is still flat.

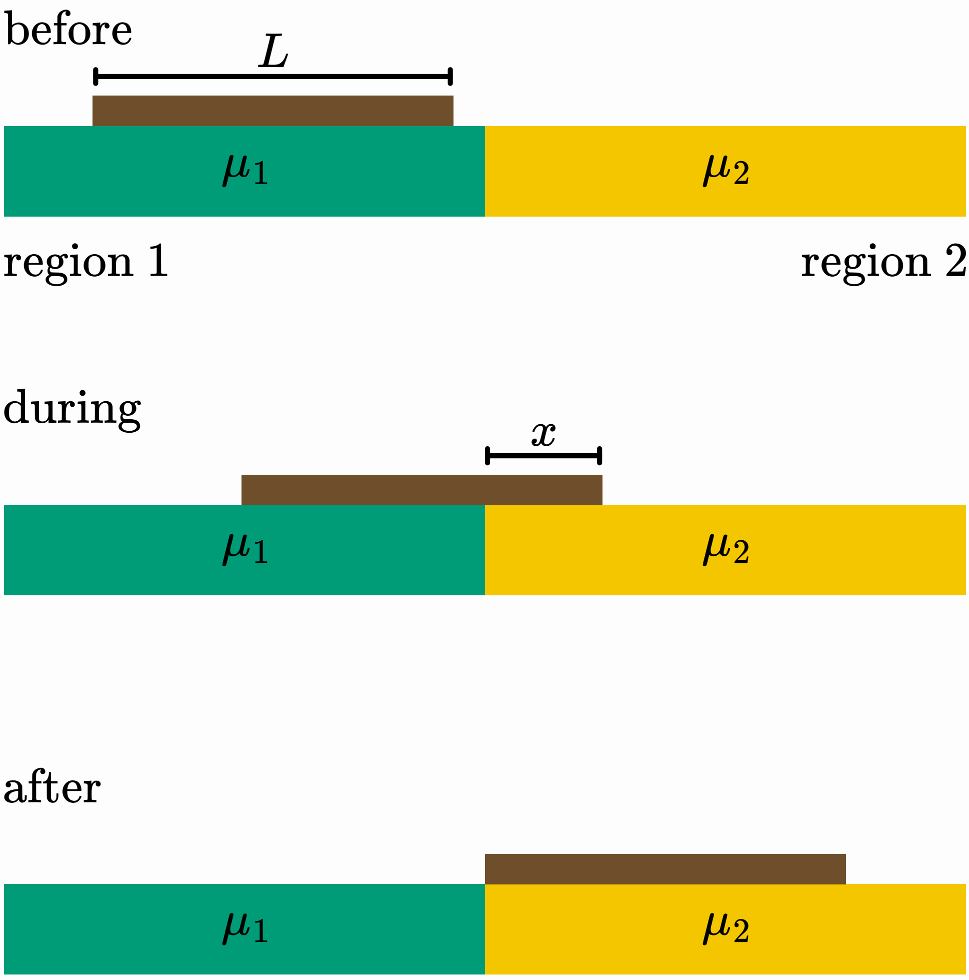
### Exercise 2.102: Pushing a box uphill[7] 🌶️

You push a box of mass $m$ up a slope with angle $\theta$ and kinetic friction coefficient $\mu$. Find the minimum initial speed $v$ you must give the box so that it reaches a height $h$. Use energy and work to find the answer.

[6]Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)
[7]Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

### Exercise 2.103: Work done dragging a board[8] 🌶️ 🌶️

A uniform board of length $L$ and mass $M$ lies near a boundary that separates two regions. In region 1, the coefficient of kinetic friction between the board and the surface is $\mu_1$, and in region 2, the coefficient is $\mu_2$. Our objective is to find the net work $W$ done by friction in pulling the board directly from region 1 to region 2, under the assumption that the board moves at constant velocity.




1. Suppose that at some point during the process, the right edge of the board is a distance $x$ from the boundary, as shown. When the board is at this position, what is the magnitude of the force of friction acting on the board, assuming that it's moving to the right? Express your answer in terms of all relevant variables ($L$, $M$, $g$, $x$, $\mu_1$, and $\mu_2$).
2. As we have seen, when the force is not constant, you can determine the work by integrating the force over the displacement, $W = \int F(x)\mathrm{d}x$. Integrate your answer from (1) to get the net work you need to do to pull the board from region 1 to region 2.

[8] Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Exercise 2.105: 🌶️ 🌶️

A point particle (mass $m$) drops from a height $H$ downwards. It starts with zero initial velocity. The only force acting is gravity (take gravity's acceleration as a constant).

- Set up the equation of motion (i.e. N2) for $m$.
- Calculate the velocity upon impact with the ground.
- Calculate the kinetic energy of $m$ when it hits the ground.
- Calculate the amount of work done by gravity by solving the integral $W_{12} = \int_1^2 \vec{F} \cdot d\vec{r}$.
- Show that the amount of work calculated is indeed equal to the change in kinetic energy.

Solve this exercise using IDEA.

Exercise 2.106: 🌶️ 🌶️

A hockey puck ($m = 160$ gram) is hit and slides over the ice-floor. It starts at an initial velocity of $10\mathrm{m/s}$. The hockey puck experiences a frictional force from the ice that can be modeled as $-\mu F_g$ with $F_g$ the gravitational force on the puck. The friction force eventually stops the motion of the puck. Then the friction is zero (otherwise it would accelerate the puck from rest to some velocity :smiley: ). Constant $\mu = 0.01$.

- Set up the equation of motion (i.e. N2) for $m$.
- Solve the equation of motion and find the trajectory of the puck.
- Calculate the amount of work done by gravity by solving the integral $W_{12} = \int_1^2 \vec{F} \cdot d\vec{r}$.
- Show that the amount of work calculated is indeed equal to the change in kinetic energy.
- Solve this exercise using IDEA.

Exercise 2.107: 🌶️

An electron (mass m, charge -e) is in a static electric field. The electric field is of the form $\vec{E} = E_0 \sin\left(2\pi \frac{X}{L}\right)\hat{x}$. As a consequence, the electron experiences a force $\vec{F} = -e\vec{E}$ Due to this force, the electron moves from position $x = \frac{1}{4}L$ to $x = 0$.

- Calculate the amount of work done by the electric field.
- Assuming that the electron was initially at rest, what is the velocity at $x = 0$?

Exercise 2.108: 🌶️

A force $F = F_0 e^{-t/\tau}$ is acting on a particle of mass m. The particle is initially at position $x = 0$. It is, starting at $t = 0$, moving at a constant velocity $v_0 > 0$ to $x = L$, $(L > 0)$.

a. Calculate the amount of work done by $F$.
b. The amount of work done is equal to the change in kinetic energy. However, the particle doesn't change its kinetic energy. Why is this general rule not violated in this case?

## Exercise 2.109: Work by a lineair force 🌶️

A point particle of mass $m = 2\text{kg}$ is at $t = 0$ at position $x = 0$. It has initial velocity $v_0$. From $t = 0$ to $t_{stop} = 4\text{s}$ it is under the influence of a force $F(x)$ that linearly increases with the position: $F(x) = kx$ with $k > 0$. This is a 1D problem.

The top graph show the position of the particle. The bottom graph shows the Work done on the particle by the force and the kinetic energy of the particle.

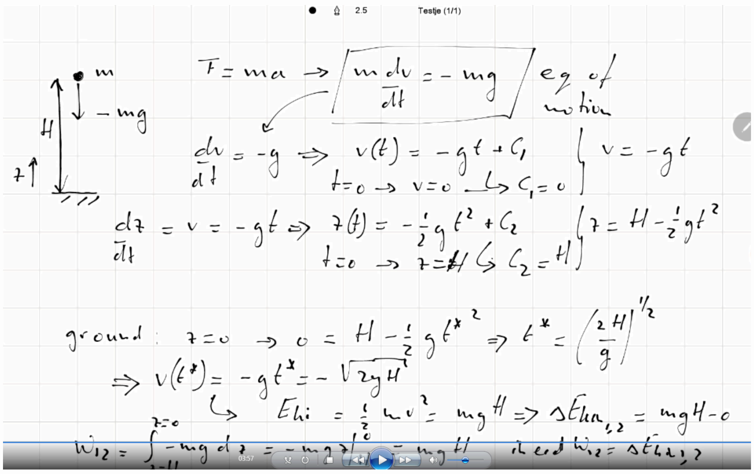
Analyse this situation and calculate the work done by the force at any time. Is the work done in this case always sufficient to account for the change in kinetic energy? What does it mean if the work is positive or negative?

Are the red ($W$) line and the green ($E_{kin}$) parallel? What does that mean?

## Solution 2.110: Solution to Exercise 9

1. $W = \Delta E_{kin} = \int_0^x F\mathrm{d}x = \int_0^x kx dx = 1/2kx^2 = 1/2mv^2 \Rightarrow v = \sqrt{\frac{kx^2}{m}}$
2. The spring has mass as well.
3. The gravitational does work as well ($W = F_g \mathrm{d}x < 0$)

## Solution 2.111: Solution to Exercise 13



Solution 2.113: Solution to Exercise 15

Work done by electric field when the electron moves from $x = \frac{1}{4}L$ to $x = 0$:

$$W = \int_{\frac{1}{4}L}^{0} \vec{F}\ .d\vec{s} = -eE_0 \int_{\frac{1}{4}L}^{0} \sin\Big(2\pi\frac{x}{L}\Big)dx = \\ -eE_0\frac{L}{2\pi}\Big[-\cos\Big(2\pi\frac{x}{L}\Big)\Big]_{\frac{1}{4}L}^{0} = \frac{1}{2\pi}eE_0L \tag{2.198}$$

Work done is gain in kinetic energy: $\Delta E_{kin} = W$. Assuming the only work done is by the electric field and using initial velocity is zero: $v_i = 0$ :

$$\frac{1}{2}mv^2 = \frac{1}{2\pi}eE_0L \Rightarrow v = \sqrt{\frac{eE_0L}{\pi m}} \tag{2.199}$$

Note that indeed the work done is positive, as it should in this case since the electron starts with zero velocity.

#### 2.3.8.1 Exercise set 1

#### 2.3.8.2 Answers set 1

#### 2.3.8.3 Exercise set 2

#### 2.3.8.4 Answers set 2

Solution 2.114: Solution to Exercise 16

$$W = \int_{0}^{L} \vec{F}\ .d\vec{s} = \int_{0}^{L} F_0e^{-\frac{t}{\tau}}\ dx \tag{2.200}$$

Particle velocity is $v_0 = const$. Thus, trajectory $x(t) = v_0t$ since at $t = 0 \rightarrow x = 0$
Consequently: $x = L \rightarrow t = \frac{L}{v_0}$

Thus, we can write for the amount of work done:

$$W = \int_{0}^{\frac{L}{v_0}} F_0e^{-\frac{t}{\tau}} \cdot v_0dt = \\ F_0v_0\Big[-\tau e^{-\frac{t}{\tau}}\Big]_{0}^{L/v_0} = F_0v_0\tau\Big(1 - e^{-\frac{L}{v_0\tau}}\Big) \tag{2.201}$$

We note: $W > 0$ and naively, we could expect that the kinetic energy of the particle would have increased. But that isn't the case: it started with $E_{kin} = \frac{1}{2}mv_0^2$ and it kept this along the entire path as it is given that the particle is traveling with a constant velocity.

From this last statement, we immediately learn, that there must be a second force acting on the particle. This force is exactly equal and opposite to $F$ at all times! Otherwise, the particle would accelerate and change its velocity. Consequently, this second force also perform work on $m$, the amount is exactly $-W$ and thus the total work done on the particle is zero which reflects that the particle does not change its kinetic energy.

## 2.4 Angular Momentum, Torque & Central Forces

Updated: 18 jan 2026

### 2.4.1 Torque & Angular Momentum

From experience we know that if we want to unscrew a bottle, lift a heavy object on one side, try to unscrew a screw, we better use ‘leverage’.

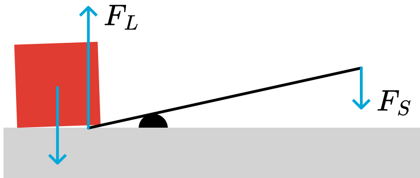



Figure 2.115: Lifting is easier using leverage.

With a relatively small force $F_S$, we can lift the side of a heavy object. The key concept to use here is torque, which in words is loosely formulated: apply the force using a long arm and the force seems to be magnified. The torque is then force multiplied by arm:

$\tau = Force\ times\ arm$

This is, of course, too sloppy for physicists. We need strict, formal definitions. So, we put the above into a mathematical definition.

**torque**

$$\vec{\tau} \equiv \vec{r} \times \vec{F} \tag{2.202}$$

That is: torque (or krachtmoment in Dutch) is the cross product of ‘arm’ as a vector(!) and the force. We notice a few peculiarities.

1. like force, torque is a vector. That is: it has a magnitude and a direction. In principle: three components.
2. its direction is perpendicular to the force vector $\vec{F}$ *and* perpendicular to the arm $\vec{r}$.
3. the arm is not a number: it is a vector!

We further know from experience that we can balance torques, like we can balance forces. Rephrased: the net effect of more than one force is found by adding all the forces (as vectors!) and using the net force in Newtons Second Law: $m\vec{a} = \sum \vec{F}_i = \vec{F}_{net}$. From Newtons First Law, we immediately infer: if $\sum \vec{F}_i = \vec{F}_{net} = 0$ then the object moves at constant velocity. We can move with the object at this speed and conclude that it from this perspective has zero velocity: it doesn’t move, i.e. it is in equilibrium.

The same holds for torque: we can work with the sum of all torques that act on an object: $\sum \vec{\tau}_i = \vec{\tau}_{net}$. And if this sum is zero, the object is in equilibrium.

However, there is a catch: using torques requires that we are much more explicit and precise about the choice of our origin. Why? The reason is in the ‘arm’. That is only defined if we provide an origin.

#### 2.4.1.1 The seesaw and torque

Let’s consider a simple example (simple in the sense that we are all familiar with it): the seesaw.

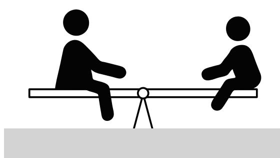


Figure 2.116: An adult (left) and a child (right) on a seesaw.

It is obvious that the adult -seesawing with the child- should sit much closer to the pivot point than the child. That is: we assume that the mass of the adult is greater than that of the child.

Let's turn this picture into one that captures the essence and includes the necessary physical quantities, and then draw a free-body diagram.

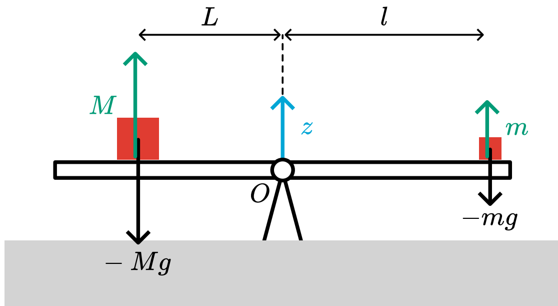



Figure 2.117: Free-body diagram of the seesaw and the masses.

What did we **draw**?

1. The force of gravity acting on the two masses $M$ and $m$. That is obvious: without forces nothing will happen and there is nothing to be analyzed.
2. The 'reaction forces' from the seesaw on both masses. Why? If the seesaw is in equilibrium, then each of the masses is in equilibrium and the sum of forces on each mass must be zero.
3. The distance of each of the masses to the pivot point. Why? Leverage! The heavy $M$ must be closer to the pivot point the get equilibrium.
4. An origin $O$. Why? We need a point to measure the 'arm', 'leverage', from.
5. The $z$-coordinate. Why? We deal with forces in the vertical direction. Hence a coordinate, a direction that we all use, is handy.

**Analysis**
Time for a first analysis: what keeps this seesaw in equilibrium?

1. The sum of forces on each of the masses is zero. As gravity pulls them down, the seesaw must exert a force of the same magnitude but in the opposite direction. These are the green forces.
2. With this idea we have the masses in equilibrium, but not necessarily the seesaw. Why? We did not consider forces on the seesaw. Which are these: (a) the reaction force (i.e. the N3 pair) of the green force from the seesaw on mass $M$. We did not draw that! Similarly, for the mass $m$.
3. Now that we focus on the seesaw itself: this is in equilibrium (that is given), but there are two forces acting on it in the negative $z$-direction as we found in (2). Even if we consider the mass of the seesaw: that will not help, gravity will pull it downwards. What did we forget? The force at the pivot point, of course! The pivot will exert an upward force, preventing the seesaw from falling down. For simplicity, we assume that the seesaw has zero mass. Thus, there are three forces acting on it: $-Mg$, $-mg$, $F_p$ with $F_p - Mg - mg = 0$.

Let's redraw, now concentrating on the forces on the seesaw.

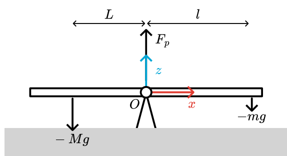



Figure 2.118: Free-body diagram of the seesaw.

**Analysis part 2**
We know that the seesaw is in equilibrium, thus

$$F_p - Mg - mg = 0 \tag{2.203}$$

This guarantees that the seesaw does not change its velocity, and as it does not move at some time $t_0$, it doesn't move for all $t > t_0$.

But this doesn't guarantee that the seesaw doesn't rotate around the pivot point. For that we need that the 'leverages' on the left and right side 'perform' the same.
Making this precise: the torques exerted on the seesaw must also equate to zero.
We have 3 forces, thus 3 torques: $-Mg$ with arm $L$, $-mg$ with arm $l$ and $F_p$ with arm zero.

Now we need to be even more precise: torque is a vector and it is made as a cross product of the vector 'arm' and the force.
We have already drawn an $x$-coordinate in the figure, that will allow us to write the 'arm' as a vector. After all, we need to evaluate the cross product $\vec{r} \times \vec{F}$. We do that for the three forces, starting on the left:

$$\vec{\tau}_1 = -L\hat{x} \times (-Mg)\hat{z} = MLg\,\hat{x} \times \hat{z} = MLg(-\hat{y}) = -MLg\,\hat{y} \tag{2.204}$$

We have used here, that the cross product of $\hat{x}$ with $\hat{z}$ is equal to $-\hat{y}$ with $\hat{y}$ the unit vector in the $y$-direction pointing into the screen.

Similarly for the force coming from the small mass $m$ on the right side:

$$\vec{\tau}_2 = l\hat{x} \times (-mg)\hat{z} = -mlg\,\hat{x} \times \hat{z} = mlg\,\hat{y} \tag{2.205}$$

Finally, the torque from the force exerted by the pivot point:

$$\vec{\tau}_3 = 0\hat{x} \times F_p\hat{z} = 0 \tag{2.206}$$

Next, we evaluate the total torque:

$$\vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 = (mlg - MLg)\hat{y} \tag{2.207}$$

In order for the seesaw not to start rotating, we must have that the torque is zero and thus:

$$\sum \vec{\tau}_i = 0 \Rightarrow mlg = MLg \rightarrow \frac{m}{M} = \frac{L}{l} \tag{2.208}$$

A result we expected: the greater mass should be closer to the pivot point.

#### 2.4.1.2 Different origin

So far, so good. But what if we had chosen the origin not at the pivot point, but somewhere to the left? Then all 'arm' will change length. And all torques will be different. Wouldn't that make $\sum \vec{\tau}_i \neq 0$?
No, it wouldn't! Let's just do it and recalculate. In the figure below, we have moved the origin to the left end of the seesaw. The distance from the heavy mass to the origin is $\Lambda$ (green arrow).

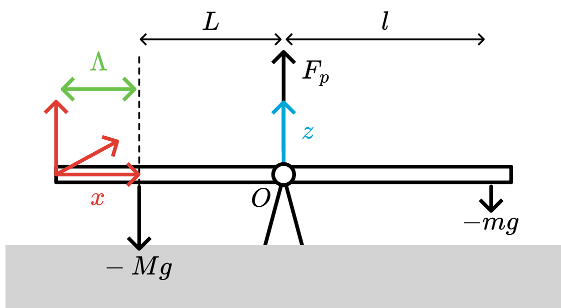



Figure 2.119: Free-body diagram with the origin located at the seesaw's end.

We still have that the sum of forces is zero. But what about the sum of torques? Obviously, the choice of the origin cannot affect the seesaw: it is still in balance, regardless of our choice of the origin. Let's see if that is correct:

$$\sum \vec{\tau}_i = \Lambda\hat{x} \times -Mg\hat{z} + (\Lambda + L) \times F_p\hat{z} + (\Lambda + L + l)\hat{x} \times -mg\hat{z} \tag{2.209}$$

We have drawn the three unit vectors $\hat{x}, \hat{y}, \hat{z}$ in the figure. We will use again: $\hat{x} \times \hat{z} = -\hat{y}$. This simplifies the torque equation above to:

$$\sum \vec{\tau}_i = \left[Mg\Lambda - (\Lambda + L)F_p + mg(\Lambda + L + l)\right]\hat{y} \tag{2.210}$$

This is clearly more complicated than the expression we had with the first choice of the origin. Moreover, the force from the pivot point shows up in our expression.

Luckily, we have equilibrium. Hence: $F_p - Mg - mg = 0 \Rightarrow F_p = Mg + mg$. We substitute this into our torque equation:

$$\begin{aligned} \sum \vec{\tau}_i &= [Mg\Lambda - (\Lambda + L)(Mg + mg)) + mg(\Lambda + L + l)]\hat{y} \\ &= [Mg(\Lambda - (\Lambda + L)) + mg(-(\Lambda + L) + \Lambda + L + l)]\hat{y} \\ &= [-MgL + mgl]\hat{y} \end{aligned} \tag{2.211}$$

Which is exactly the same expression as we found before. So, indeed, the choice of the origin doesn't matter.

**Conclusion**

For equilibrium we need that the sum of torques is zero:

$$\sum_i \vec{\tau}_i = 0 \tag{2.212}$$

### 2.4.2 Angular Momentum

From our seesaw example we learn: the seesaw can only be in equilibrium if the sum of torques is zero. What if this sum is non-zero? That is, a net torque operates on the seesaw.

We know that the seesaw will rotate and in order to balance it, we have to shift at least one of the masses.

In which direction will it rotate?

Before answering: first we need to think about **direction of rotation**. Does it have direction and if so: how do we make clear what that is?

Again the seesaw will give guidance. Suppose we remove the smaller mass all together. Then, it is obvious: the 'heavy' left side will rotate to the ground and the light right side upwards. From the point of view we are standing: the seesaw will rotate counter clockwise.

We will use the corkscrew rule or right hand rule to give that a direction: rotate a corkscrew clockwise and the screw will move into the cork away from you; rotate a corkscrew counter clockwise and it will move out of the cork, towards you. Of course, in stead of a corkscrew you can think of a screwdriver or a water tap: closing is rotating 'clock wise, opening the tap is counter clockwise.

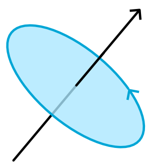


Figure 2.120: The rotation is given by the black arrow. You can find it by using the corkscrew rule: rotating a corkscrew as the blue arrow indicates gives that the screw moves forward like the black arrow.

With this, we can define the direction of rotation better than via clock or counter clock wise. In our seesaw example, we will say: if the seesaw rotates clockwise, its rotation is described by a vector that points in the positive $y$-direction as given in the figure, that is pointing into the paper (or screen).

Now, we can couple this to the direction of the torque. We saw -taking the origin at the pivot point- two torques acting on the seesaw. The large mass has its torque pointing in the negative $y$-direction: it points out of the screen/paper. And this torque tries to rotate the seesaw counter clockwise. On the other hand, the small mass has a torque pointing in the positive $y$-direction which is in line with it trying to rotate the seesaw clockwise.
Which of the two is 'strongest' determines the direction of rotation: if $MgL > mgl$ then the net torque is in the minus-$y$ direction. That is, the torque of the larger mass is more negative than the smaller one is positive: $-MgL + mgl < 0$ and the net torque points towards us.

The quantity that goes with this, is the angular momentum. It is defined as

**angular momentum**

$$\vec{l} \equiv \vec{r} \times \vec{p} \tag{2.213}$$

Note that it is a cross product as well. Hence it is a vector itself. Further note that $\vec{r} \times \vec{p} \neq \vec{p} \times \vec{r}$. The order matters! First $\vec{r}$ then $\vec{p}$. If you do it the other way around, you unwillingly have introduced a minus sign that should not be there.
Furthermore, note that since $\vec{l} \equiv \vec{r} \times \vec{p}$, $\vec{l}$ is perpendicular to the plane formed by $\vec{r}$ and $\vec{p}$.

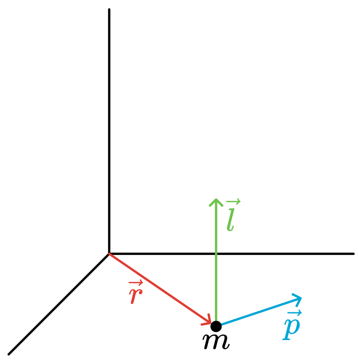



Figure 2.121: Angular momentum of a particle at a certain position with momentum.

#### 2.4.2.1 Torque & Analogy to N2

Angular momentum obeys a variation of Newton's second law that ties it directly to torque.

$$\vec{l} = \vec{r} \times \vec{p} \Rightarrow \tag{2.214}$$

$$\frac{d\vec{l}}{dt} = \frac{d(\vec{r} \times \vec{p})}{dt} = \underbrace{\underbrace{\frac{d\vec{r}}{dt}}_{=\vec{v}} \times \vec{p}}_{=0 since \vec{v}//\vec{p}} + \vec{r} \times \underbrace{\frac{d\vec{p}}{dt}}_{N2:=\vec{F}} = \vec{r} \times \vec{F} \tag{2.215}$$

Thus, we find a general law for the angular momentum:

**N2 for angular momentum**

$$\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} \tag{2.216}$$

Again, note that the right hand side is a cross product, so the order does matter.

With the torque denoted by $\vec{\tau}$, we have

$$\vec{\tau} \equiv \vec{r} \times \vec{F} \tag{2.217}$$

then we can write down an equation similar to N2 $\left(\dot{\vec{p}} = \vec{F}\right)$ but now for angular motion

$$\dot{\vec{l}} = \vec{\tau} \tag{2.218}$$

where the force is replaced by the torque and the linear momentum by the angular momentum.

NB: Note that the torque and angular moment change if we choose a different origin as this changes the value of $\vec{r}$.

**Intermezzo: cross product**

Here is some recap for the cross product. See also the Lin. Alg. book page. A cross product of two vectors $\vec{a}$ and $\vec{b}$ is defined as

$$\vec{c} = \vec{a} \times \vec{b} \equiv |\ a\ |\ |\ b\ | \sin\theta\, \hat{n} \tag{2.219}$$

Here $\theta$ is the angle between $\vec{a}$ and $\vec{b}$, and $\hat{n}$ is a unit vector normal to the plane spanned by $\vec{a}, \vec{b}$ with direction given by the *right-hand rule*.

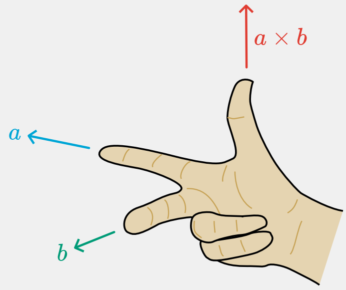



Figure 2.122: Right hand rule for cross products. Adapted from Wikimedia Commons, licensed under CC-BY-SA 4.0.

From the definition it is clear that $|\ \vec{a} \times \vec{b}\ |$ is the area of the parallelogram spanned by $\vec{a}, \vec{b}$.

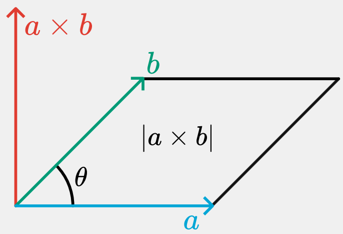



Figure 2.123: Area of cross products. From Wikimedia Commons, public domain.

The cross product is bilinear, anti commutative $\left(\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)\right)$ and distributive over addition.

The formula is for computation in an orthonormal basis is

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -(a_1 b_3 - a_3 b_1) \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \tag{2.220}$$

Figure 2.124: How to *remember* this rule for the cross product in Cartesian or polar coordinates.

The formula can be derived from the cross product for orthonormal basis vectors, e.g. $\hat{x}, \hat{y}, \hat{z}$

$$\begin{aligned} \hat{x} \times \hat{y} &= \hat{z} \\ \hat{y} \times \hat{z} &= \hat{x} \\ \hat{z} \times \hat{x} &= \hat{y} \end{aligned} \tag{2.221}$$

Notice the cyclic structure of the equations.

It is a common mistake to identify angular momentum with rotational motion. That is not correct. A particle that travels in a straight line will, in general, also have a non-zero angular momentum, see Figure 11. Here we look at a free particle: there are no forces working on it. So it travels in a straight line, with constant momentum.

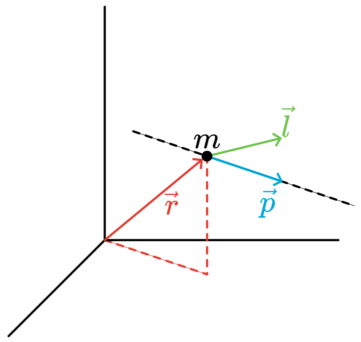



Figure 2.125: Angular momentum of a free particle.

However, the particle position does change over time. So: is its angular momentum constant or not?
That is easy to find out. We could take 'N2' for angular momentum:

$$\frac{d\vec{l}}{dt} = \vec{r} \times \underbrace{\underbrace{\vec{F}}_{=0}}_{freeparticle} = 0 \Rightarrow \vec{l} = const \tag{2.222}$$

Clearly, the angular momentum of a free particle is constant. Moreover, the momentum of a free particle is also constant. But what about the position vector: isn't that changing over time and eventually becomes very, very long? Why does that not change $\vec{r} \times \vec{p}$?
Take a look at Figure 12. We have chosen the $xy$-plane such that both $\vec{r}$ and $\vec{p}$ are in it. Furthermore, we have taken it such that $\vec{p}$ is parallel to the $x$-axis.

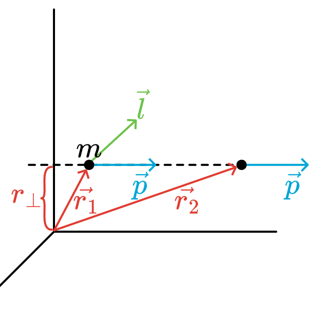



Figure 2.126: Angular momentum of a free particle is constant.

At some point in time, the particle is at position $\vec{r}_1$. Its angular momentum is perpendicular to the $xy$-plane and has magnitude $|\,|\vec{r}_1 \times \vec{p}\,|\,| = r_\perp p$. Later in time it is at position $\vec{r}_2$. Still, its angular momentum is perpendicular to the $xy$-plane and has magnitude $|\,|\vec{r}_2 \times \vec{p}\,|\,| = r_\perp p$, indeed identical to the earlier value. This shows that indeed the angular momentum of a free particle is constant.

### 2.4.3 Examples & Exercises

**Example: Throwing a basketball**

As seen in class: one person throws a basketball to another via a bounce on the ground, the basketball starts to spin after hitting the ground although initially it did not.

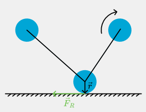


Figure 2.127: A bouncing basketball.

When the ball hits the ground a friction force is acting on the ball. This force will apply a torque on the ball. The friction is directed opposite to the direction of motion. The arm $\vec{r}$ from the center of the ball to where the force is acting, is downwards. Using the right-hand rule we find that the torque is pointing in the plane of the screen, and thus the rotation is clockwise (forwards spin).

The forwards momentum of the ball is reduced by the action of the force. The upwards components is just flipped by the bounce on the ground. Therefore the outgoing ball is bouncing up at a steeper angle than it is was incoming.

**Conservation of angular momentum & spinning wheel**

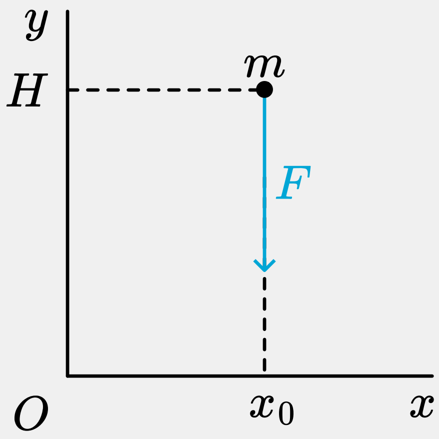
As seen in class, we have a student sitting on a chair that can rotate (swivel chair). The student is holding a bicycle wheel in horizontal position.

Figure 2.128: Student with a rotating wheel on a swivel chair.

Once the student starts to spin the wheel while sitting on the chair, the student will start to rotate in the opposite direction (with smaller angular velocity, later on we will see why their speeds are different). There is no external force on the student + wheel. Consequently, the total angular momentum must stay constant. But the student exerts an angular momentum on the wheel, causing it to rotate. But at the same time, due to action = - reaction, the wheel exerts also a torque on the student. But in the opposite direction. Thus, to compensate the angular momentum pointing up (counter clockwise rotation), an angular momentum pointing down (clockwise rotation) of the same magnitude must occur, keeping the total angular momentum of student + wheel constant.

**Note**

Exercise 1: A point particle (mass $m$) is initially located at position $P = (x_0, H, 0)$. At $t = 0$, it is released from rest and falls in a force field of constant acceleration $\vec{a} = (0, -a, 0)$ that acts on the mass.




Analyze what happens to the angular momentum of $m$.

**Note**

Exercise 2: The same question, but now the particle has an initial velocity $\vec{v} = (v_0, 0, 0)$.

**Note**

Exercise 3: Similar situation: can you find an example of a falling object for which the angular momentum stays constant? Ignore friction with the air.

**Solution to Exercise 1: A point particle (mass**

The particle falls under a force that points in the negative $y$-direction. As a consequence, it will start moving vertically downwards:

$$\begin{aligned} x &: h(1cm)m\frac{dv_x}{dt} = 0 \rightarrow v_x = C_1 = 0 \\ y &: h(1cm)m\frac{dv_y}{dt} = -ma \rightarrow v_y = -at + C_2 = -at \end{aligned} \tag{2.223}$$

Thus, we find for the momentum om the particle: $\vec{p} = (0, -m\,at)$.

The position of $m$ follows from:

$$\begin{aligned} x &: h(1cm)\frac{dx}{dt} = v_x = 0 \rightarrow x(t) = C_3 = x_0 \\ y &: h(1cm)\frac{dy}{dt} = v_y = -at \rightarrow y(t) = -\frac{1}{2}at^2 + C_4 = H - \frac{1}{2}at^2 \end{aligned} \tag{2.224}$$

We can now compute the angular momentum:

$$\begin{aligned} \vec{l} &= \vec{r} \times \vec{p} \\ &= \left(x_0\hat{x} + \left(H - \frac{1}{2}at^2\right)\hat{y}\right) \times (-mat\hat{y}) \\ &= -mx_0at\underbrace{\hat{x} \times \hat{y}}_{=\hat{z}} + x_0\left(H - \frac{1}{2}at^2\right)\underbrace{\hat{y} \times \hat{y}}_{=0} \\ &= -mx_0at\,\hat{z} \end{aligned} \tag{2.225}$$

Thus, the angular momentum is pointing in the negative $z$-direction and increases linearly with time in magnitude.

We could have tried to find this via the variation of N2 for angular momentum. Now, we need to compute the torque on $m$ and solve $\frac{d\vec{l}}{dt} = \vec{\tau}$. This goes as follows:

$$\begin{aligned}\vec{\tau} &= \vec{r} \times \vec{F} \\ &= (x\hat{x} + y\hat{y}) \times -ma\hat{y} \\ &= -ma\,x\hat{z}\end{aligned} \tag{2.226}$$

And thus:

$$\frac{d\vec{l}}{dt} = -ma\,x\hat{z} \tag{2.227}$$

To get any further, we need information about $x(t)$. From the $x$-component of N2 we know (see above): $x(t) = x_0$. If we plug this in, we get:

$$\frac{d\vec{l}}{dt} = -ma\,x_0\hat{z} \to \vec{l} = -mx_0at + C_5 = -mx_0at \tag{2.228}$$

where we have used: $t = 0 \to \vec{p} = 0 \to \vec{l} = 0 \Rightarrow C_5 = 0$

**Solution to Exercise 2: The same question, but now the particle has an initial velocity**

We can follow the same procedure as in exercise (6.1). But now, the outcome of the $x$-component of N2 changes.

$$\begin{aligned}x &: h(1cm)m\frac{dv_x}{dt} = 0 \to v_x = C_1 = v_0 \\ y &: h(1cm)m\frac{dv_y}{dt} = -ma \to v_y = -at + C_2 = -at\end{aligned} \tag{2.229}$$

Thus, we find for the momentum om the particle: $\vec{p} = (mv_0, -mat)$.

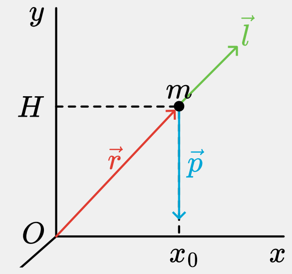
The position of $m$ follows from:

$$\begin{aligned}x &: h(1cm)\frac{dx}{dt} = v_x = v_0 \to x(t) = v_0t + C_3 = x_0 + v_0t \\ y &: h(1cm)\frac{dy}{dt} = v_y = -at \to y(t) = -\frac{1}{2}at^2 + C_4 = H - \frac{1}{2}at^2\end{aligned} \tag{2.230}$$

We can now compute the angular momentum:

$$\begin{aligned}\vec{l} &= \vec{r} \times \vec{p} \\ &= \left((x_0 + v_0t)\hat{x} + \left(H - \frac{1}{2}at^2\right)\hat{y}\right) \times (mv_0\hat{x} - mat\hat{y}) \\ &= -m(x_0 + v_0t)at\underbrace{\hat{x} \times \hat{y}}_{=\hat{z}} + \left(H - \frac{1}{2}at^2\right)mv_0\underbrace{\hat{y} \times \hat{x}}_{=\widehat{-z}} \\ &= -m\left(Hv_0 + x_0at + \frac{1}{2}v_0at^2\right)\hat{z}\end{aligned} \tag{2.231}$$

Thus, the angular momentum still points in the negative $z$-direction but increases quadratically with time in magnitude.




### 2.4.4 Central Forces

We have looked at a specific class of forces: the conservative ones. Here we will inspect a second class, that is very useful to identify: the central forces.

A force is called a central force if:

$$\vec{F} = \mid \vec{F}(\vec{r}) \mid \hat{r} \tag{2.232}$$

In words: a force is central if it points always into the direction of the origin or exactly in the opposite direction. The reason to identify this class is in the consequences it has for the angular momentum.

Suppose, a particle of mass $m$ is subject to a central force. Then we can immediately infer that its angular momentum is a constant:

$$if \vec{F} = \mid \vec{F}(\vec{r}) \mid \hat{r} then \frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} = \mid \vec{F}(\vec{r}) \mid \vec{r} \times \hat{r} = 0 \tag{2.233}$$

where we have used that $\vec{r}$ and $\hat{r}$ are always parallel so their cross-product is zero.

The above is rather trivial, but has a very important consequence: a particle that moves under the influence of a central force, moves with a constant angular momentum (vector!) and must move in a plane. It cannot get out of that plane. Thus its motion is at maximum a 2-dimensional problem. We can always use a coordinate system, such that the motion of the particle is confined to only two of the three coordinates, e.g. we can choose our $x, y$ plane such that the particle moves in it and thus always has $z(t) = 0$.

Why is this so? Why does the fact that the angular momentum vector is a constant immediately imply that the particle motion is in a plane? The argumentation goes as follows.
Imagine a particle that moves under the influence of a central force. At some point in time it will have position $\vec{r}_0$ and momentum $\vec{p}_0$. Neither of them is zero. We will assume that $\vec{r}_0$ and $\vec{p}_0$ are not parallel (in general they will not be). Thus they define a plane. Due to the cross-product $\vec{l}_0 = \vec{r}_0 \times \vec{p}_0$ is perpendicular to this plane.
A little time later, say $\Delta t$ later, both position and momentum will have changed. Since the force is central, the force is also in the plane defined by the initial position and momentum. Thus the change of momentum is in that plane as well: $\vec{p}(t + \Delta t) = \vec{p}(t) + \vec{F}\Delta t$. The right hand side is completely in our plane. And thus, the new momentum is also in the plane. But that means that the velocity is also in the same plane. An thus the new position $\vec{r}(t + \Delta t) = \vec{r}(t) + \frac{\vec{p}}{m}\Delta t$ must be in the same plane as well. We can repeat this argument for the next time and thus see, that both momentum and position cannot get out of the plane. This is, of course, fully in agreement with the fact that $\vec{l} = const$ for a central force.

### 2.4.5 Central forces: conservative or not?

We can further restrict our class of central forces:

$$if \vec{F}(\vec{r}) = f(r)\hat{r} then F is central and conservative \tag{2.234}$$

In the above, $\mid \vec{F}(\vec{r}) \mid = f(r)$, that is: *the magnitude of the force only depends on the distance from the origin not on the direction.* **Rephrased**: *the force is spherically symmetric.* If that is the case, the force is automatically conservative and a potential does exist.

Both the concept of central forces and potential energy play a pivotal role in understanding the motion of celestial bodies, like our earth revolving the sun. The planetary motion is an example of using the concept of central forces. It is, however, also an example in its own right. Using his new theory, Newton was able to prove that the motion of the earth around the sun is an ellipsoidal one. It helped changing the way we viewed the world from geo-centric to helio-centric.

#### 2.4.5.1 Keppler's Laws

Before we embark at the problem of the earth moving under the influence of the sun's gravity, we will go back in time a little bit.

**Intermezzo: Tycho Brahe & Johannes Kepler**

We find ourselves back in the Late Renaissance, that is around 1550-1600 AD. In Europe, the first signs of the scientific revolution can be found. Copernicus proposed his heliocentric view of the solar system. Galilei used his telescope to study the planets and found further evidence for the heliocentric idea. In Denmark, Tycho Brahe (1546-1601) made astronomical observations with data of unprecedented precision. He did so without the telescope as the first records of telescopes date back to around 1608 AD.


Figure 2.131: left:Tycho Brahe (1546-1601) - right: Sophia Brahe (1559-1643). From Wikimedia Commons (L, R), public domain.

Brahe initially studied law, but developed a keen interest in astronomy. He was heavily influenced by the solar eclipse of August $21^{st}$ in 1560. The eclipse had been predicted via the theory of celestial motion at that time. However, the prediction was off by a day. This led Brahe to the conclusion that in order to advance celestial science, many more and much better observations were needed. He devoted much of his time in achieving this. One of his best assistants was his younger sister, Sophie.

On November $11^{th}$ 1572, Brahe observed a bright, new star in the constellation Cassiopeia (it consists of five bright stars forming a M or W). That was another event that made him decide to spend his days (or rather nights 😊 ) gathering astronomical data. The general believe in those days was still that everything beyond the Moon was eternal, never changing. So, this new star, that all in a sudden appeared, must be closer to the earth than the Moon itself. Brahe set out to measure its daily parallax against the five stars of Cassiopeia. But he didn't observe any parallax. Consequently, the new star's position had to be farther out than the Moon and the other planets that did show daily parallax. Moreover, Brahe kept measuring for months and still found no parallax. That meant that this new star was even further out than the known planets that show no daily parallax but did so for periods of month. Brahe reached the conclusion that this new 'thing' thus could not be yet another planet, but that it was a star. Another nail to the coffin of the Aristotle view. Brahe wrote a small book about it, called De Nova Stella (published in 1573). He uses the term 'nova' for a new star. We see this back in our name for the phenomenon observed by Brahe: we call it a supernova. By now it is known that this new star, this supernova is some 8,000 light years away from us. Brahe was upset by those who denied the new findings. In his introduction of De Nova Stella he writes (given here in our modern words): “Oh, coarse characters. Oh, blind spectators of heaven”. The work and the booklet made his name in Europe as a leading scientist and astronomer.

In the winter of 1577-1578 a comet, known as the “Great Comet” appeared in the skies. It was observed by many all over the globe (from the Aztecs in the America’s via European researchers to the Arabic world, India all the way to Japan). Brahe made thousands of recordings, some simultaneously done in Denmark (close to Copenhagen) and Prague. That way, Brahe could establish that the comet was much beyond the Moon.

At the end of his life, Brahe moved to Prague to become the official imperial astronomer under the protection of Rudolf II, the Holy Roman Emperor. In the later part of his life, Brahe had Johannes Kepler as his assistant.

Kepler was 6 years old when the Great Comet appeared in the sky. He recorded in his writings that his mother had taken him to a high place to look at it. At the age of nine, he witnessed a lunar eclipse in which the Moon is in the Earth shadow, darkening it and turning quite red. As a child he suffered from smallpox making his vision weak and limited ability to use his hands. This made it difficult for him to make astronomical observations. It pushed him to mathematics. But there he was confronted with the Ptolemaic and the Copernican view on planetary motion. Kepler became a math professor at the Protestant Stiftsschüle in Graz. He wrote his ideas about the universe, following the thoughts of Copernicus in a book, that was read by Tycho Brahe. This brought him into contact with Brahe. In 1600 Kepler and his family moved to Prague as a consequence of political and religious oppression. He was appointed as assistant to Brahe and worked with Brahe on a new star catalogue and planetary tables. Brahe died unexpectedly on October 24th 1601. Two days later, Kepler was appointed as his successor.

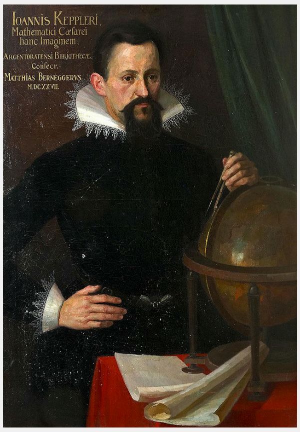



Figure 2.132: Johannes Kepler (1571-1630). From Wikimedia Commons, public domain.

Kepler worked on a heliocentric version of the universe and in the period 1609-1619 published his first two laws. With these, he changed from trying circular orbits to other closed ones, to arrive at an elliptical one for Mars. That one was in very good agreement with the Brahe data, much better than had been achieved before. Kepler realized that the other planets might also be in elliptical orbits. In comparison with Copernicus he stated: the planetary orbits are not circles with epi-circles. Instead it are ellipses. Secondly, The sun is not at the center of the orbit, but in one of the focal points of the ellipse. Thirdly, the speed of a planet is not a constant.

Kepler’s work was not immediately recognized. On the contrary, Galilei completely ignored it and many criticized Kepler for introducing physics into astronomy.

Kepler has formulated three laws that describe features of the orbits of the planets around the sun.

1. The orbit of a planet is an ellipse with the Sun at one of the two focal points.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time (Law of Equal Areas).
3. The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.

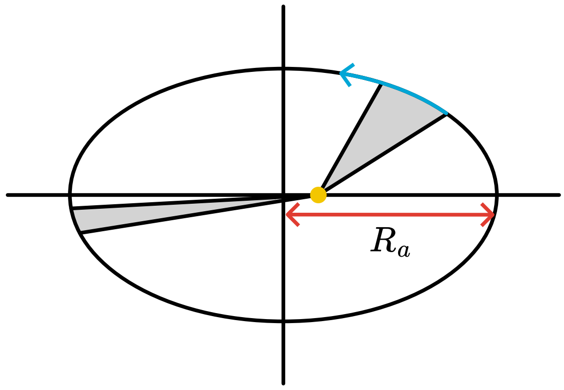



Figure 2.133: Kepler's 2nd Law of Equal Area.

$$\frac{T_A^2}{R_A^3} = \frac{T_B^2}{R_B^3} = const. \tag{2.235}$$

```
interactive(children=(IntSlider(value=1, description='t_sim', max=27374,
step=1825), Output()), _dom_classes=(…
```

```
<function __main__.sim_kep(t_sim)>
```

It is important to realize, that Kepler came to his laws by -what we would now call- curve fitting. That is, he was looking for a generic description of the orbits of planets that would match the Brahe data. He abandoned the Copernicus idea of circles with epi-circles with the sun in the center of the orbit. Instead he arrived at ellipses with the sun out of the center, in one of the focal points of the ellipse.

But, there was no scientific theory backing this up. It is purely 'data-fitting'. Nevertheless, it is a major step forward in the thinking about our universe and solar system. It radically changed from the idea that the universe is 'eternal', that is for ever the same and build up of circles and spheres: the mathematical objects with highest symmetry showing how perfect the creation of the universe is.

Kepler had formulated his laws by 1619 AD. It would take another 60 years before Isaac Newton showed that these laws are actually imbedded in his first principle approach: all that is needed is Newton's second law and his Gravitational Law.

### 2.4.6 Newton's theory and Kepler's Laws

The planets move under the influence of the gravitational force between them and the sun. We start with inspecting and classifying the force of gravity. Newton had formulated the Law of gravity: two objects of mass $m_1$ and $m_2$, respectively, exert a force on each other that is inversely proportional to the square of the distance between the two masses and is always attractive. In a mathematical equation, we can make this more precise:

$$\vec{F}_g = -G\frac{m_1 m_2}{r_{12}^2}\hat{r}_{12} \tag{2.236}$$

In the figure below, the situation is sketched. We have chosen the origin somewhere and denote te position of the sun and the planet by $\vec{r}_1$ and $\vec{r}_2$. Gravity works along the vector $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$. The corresponding unit vector is defined as $\hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}}$.

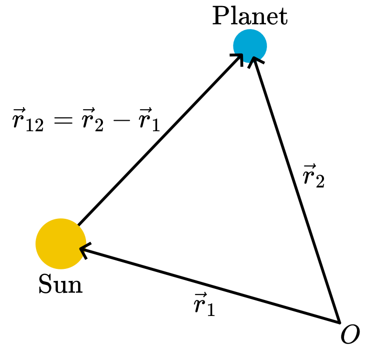



Figure 2.134: The sun and a planet.

Newton realized that he could make a very good approximation. Given that the mass of the sun is much bigger than that of a planet, the acceleration of the sun due to the gravitational force of the planet on the sun is much less than the acceleration of the planet due to the sun's gravity. For this, we only need Newton's 3rd law:

$$F_{g,sunonplanet} = -F_{g,planetonsun} \tag{2.237}$$

Hence

$$m_{sun}a_{sun} = -m_{planet}a_{planet} \to a_{sun} = \frac{m_{planet}}{m_{sun}}a_{planet} \ll a_{planet} \tag{2.238}$$

Newton concluded, that for all practical purposes, he could treat the sun as not moving. Next, he took the origin at the position of the sun. And from here on, we can ignore the sun and pretend that the planet feels a force given by

$$\vec{F}(\vec{r}) = -G\frac{mM}{r^2}\hat{r} \tag{2.239}$$

with $M$ the mass of the sun and $m$ that of the planet. $r$ is now the distance from the planet to the origin and $\hat{r}$ the unit vector pointing from the origin to the planet.

**First observation:** The force is central!

**First conclusion:** Then the angular momentum of the planet is conserved (is a constant during the motion of the planet) and the motion is in a plane, i.e. we deal with a 2-dimensional problem!

**Second Observation:** The force is of the form $\vec{F}(\vec{r}) = f(r)\hat{r}$

**Second conclusion:** Thus, we do know that a potential energy can be associated with it. It is a conservative force. This also implies that the mechanical energy of the planet, that is the sum of kinetic en potential energy, is a constant over time. In other words, there is no frictional force and the motion can continue forever. This seems to be inline with our observation of the universe: the time scales are so large that friction must be small.

#### 2.4.6.1 Constant Angular Momentum: Equal Area Law

The first clue towards the Kepler Laws comes from angular momentum. Let's consider the earth-sun system (ignoring all other planets in our solar system). As we saw above, gravity with the sun pinned in the origin, is a central force and thus

$$\frac{d\vec{l}}{dt} = \vec{r} \times \left(-G\frac{mM}{r^2}\frac{\vec{r}}{r}\right) = 0 \tag{2.240}$$

Thus, $\vec{l} = const.$ both in length and in direction. From the latter, we can infer that the motion of the earth around the sun is in a plane. Hence, we deal with a 2-dimensional problem.

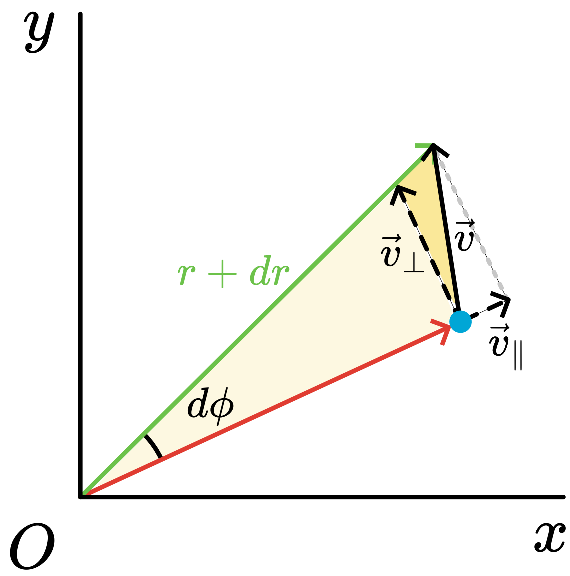



Figure 2.135: A free body diagram to help determine the area.

At some point in time, the earth is at location $\vec{r}$ (see red arrow in Figure 21). It has velocity $\vec{v}$, given by the black arrow. In a small time interval $dt$, the earth will move a little and arrive at $\vec{r} + d\vec{r}$ (the green arrow). As the time interval is very short, we can treat the velocity as a constant and thus write: $d\vec{r} = \vec{v}dt$.

Instead of concentrating on the motion of the earth, we focus on the area traced out by the earth orbit in the interval $dt$. That is the yellow area in Figure 21. This area is composed of two parts: the light yellow part and a smaller, bright yellow part. The light yellow part has an area $A_1 = \frac{1}{2} height x base$. If we make $dt$ very small, the height is almost equal to $r$ and the base becomes $v_\perp dt$ and thus $A_1 \approx \frac{1}{2} r v_\perp dt$. For the smaller yellow triangle we have: $A_2 = \frac{1}{2} dr x base \approx \frac{1}{2}\left(v_{//} dt\right) \cdot (v_\perp dt) = \frac{1}{2} v_{//} v_\perp dt^2$.

The total area traced out by the earth orbit during $dt$ is thus in good approximation:

$$dA = A_1 + A_2 = \frac{1}{2}\left(r v_\perp + v_{//} v_\perp dt\right) dt \tag{2.241}$$

We divide both sides by $dt$ and take the limit $dt \to 0$:

$$\frac{dA}{dt} = \left(\frac{1}{2} r v_\perp + \frac{1}{2} v_\perp v_/ / dt\right) \to \frac{1}{2} r v_\perp \tag{2.242}$$

In stead of $v_\perp$ we can also write $\frac{p_\perp}{m}$:

$$\frac{dA}{dt} = \frac{1}{2m} r p_\perp \tag{2.243}$$

But $r p_\perp$ is the magnitude of $\vec{r} \times \vec{p}$. And that is the magnitude of the angular momentum: $l = | \, | \, \vec{r} \times \vec{p} \, | \, | = r p_\perp$!!!

We know $l$ is constant, thus we have found:

$$\frac{dA}{dt} = \frac{1}{2m} r p_\perp = \frac{l}{2m} = constant \tag{2.244}$$

We can easily integrate this equation:

$$\frac{dA}{dt} = \frac{l}{2m} \to A(t) = \frac{l}{2m} t + C \tag{2.245}$$

We can set the constant $C$ to zero at some point in time $t_0$ and start counting the increase of the swept area. But we immediately infer that if we check the swept area between $t$ and $t + \Delta t$, this will be $\Delta A = \frac{l}{2m}\Delta t$ regardless of where the earth is in its orbit. In words: in equal time intervals, the earth sweeps an area that is the same for any position of the earth. We have established the Equal Area Law!

#### 2.4.6.2 Newton's theory and Kepler's Laws - part 2

We have:

- The sun is replaced by a force field originating at the origin. This force field is a central force.
    1. Thus, the angular momentum is conserved.
    2. The orbit is in a plane: we deal with a 2-dimensional problem.
- The force is conserved: a potential exists.

Based on these, we will derive Kepler's laws only using Newtonian Mechanics. This is easiest in polar coordinates $(r, \phi)$. However, in this course we do not deal with these coordinates. We will thus give a coarse overview of the steps that should be taken.

The first thing we notice, is that the constant angular momentum provides a constraint on the relation between $\vec{r}$ and $\vec{p}$. This constraint can be used to rewrite the kinetic energy $E_{kin} = \frac{1}{2}mv^2$ into $E_{kin} = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2}$.

What does this mean? The coordinate $r$ is the distance from the sun to the earth. Its time derivative $\left(\dot{r} = \frac{dr}{dt} = v_r\right)$ is the velocity of the earth away from the sun. This is called the radial component of the velocity. Figure 22 illustrates this.

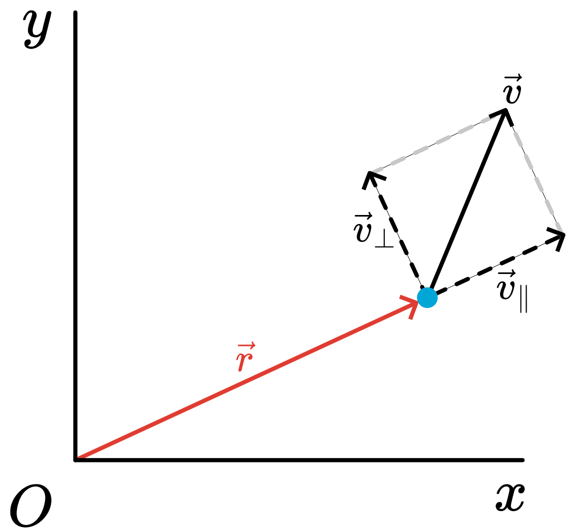



Figure 2.136: The coordinate $r$ is the distance from the sun to the earth. Its time derivative $\left(\dot{r} = \frac{dr}{dt} = v_r\right)$ is the velocity of the earth away from the sun.

It is important to realize that $\dot{r}$ tells us if we are moving such that we are getting closer to the sun or further away. But it does not tells us how we move 'around' the sun. For that we need the information of the component of the velocity perpendicular to $\vec{r}$ (the other dashed vector in the figure).

So, it seems that we are working with incomplete information. And in a sense we do. But it will turn out to be very useful to understand the physics of the earth's orbit.

We already saw that in this case gravity is a conservative force. The potential energy is found by solving $V(r) = -\int_{r_{ref}}^{r} \vec{F}_g \cdot d\vec{r}$. We can plug in $\vec{F}_g = -G\frac{mM}{r^2}\hat{r}$. Thus only the radial coordinate is of importance in the inner product in the integral. Furthermore, we will use as reference boundary: $\infty$. Thus, the potential energy is:

$$
\begin{aligned}
V(r) &= -\int_{r_{ref}}^{r} \vec{F}_g \cdot d\vec{r} \\
&= GmM \int_{\infty}^{r} \frac{dr}{r^2} \\
&= -G\frac{mM}{r}
\end{aligned} \tag{2.246}
$$

Thus, energy conservation can be written as:

$$
\frac{1}{2}m(v_x^2 + v_y^2) - -G\frac{mM}{r} = E_0 = const \tag{2.247}
$$

As expected: we have an equation with two unknowns $(x(t), y(t))$. Once we solved the problem, we will thus have the coordinates of the planet's trajectory as a function of time. However, we will not do that. Reason: it is complicated and we don't need it! What we need is to find what kind of figure the trajectory is.

Our first step is to bring the number of unknowns in the energy equation down from two to one. For that, we use $E_{kin} = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2}$.

$$
\frac{1}{2}\dot{r}^2 + \frac{l^2}{2mr^2} - -G\frac{mM}{r} = E_0 = const \tag{2.248}
$$

Now we have an equation with only one unknown $r(t)$.

We can interpret this in a different way: the second term, with the angular momentum, originates from kinetic energy, but now looks like a potential energy. And that is exactly what we are going to do: treat it as a potential energy.

Hence, we can first inspect the global features of our energy equation. Notice that the gravity potential energy is an increasing function of the distance from the planet to the sun (located and fixed in the origin). This shows that the underlying force attractive is. The new part, coming from angular momentum, on the other hand is a decreasing function of distance. Thus, the related force is repelling.

We can make a drawing of the energy. See Figure 23.

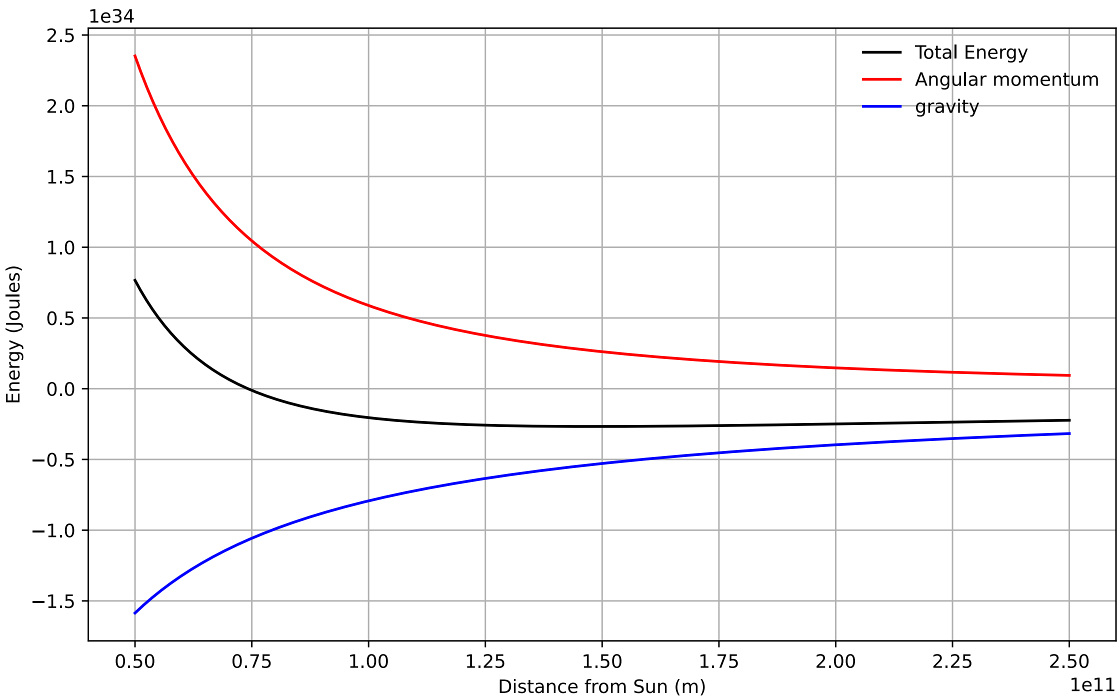



Figure 2.137: Energies related to our planet, with a minimum around $1.5e11m$.

The blue line is the potential energy of gravity. The red one stems from the kinetic energy associated with the angular velocity. The black line is the sum of the two, a kind of effective potential:

$$U_{eff} = \frac{l^2}{2mr^2} - -G\frac{mM}{r} \tag{2.249}$$

We see, that the energy cannot be just any value: the kinetic energy of our quasi-one-dimensional particle ($\frac{1}{2}m\dot{r}^2$) cannot be negative and the total potential energy has, according to Figure 23 a clear minimum. The total energy cannot be below this minimum. On the other hand: there is no maximum.

**Case 1: Effective potential = minimal**

Suppose, we would prepare the system such that its total energy was equal to the minimum of the black line, i.e. of the total potential energy. Then, of course, via the arguments we have given above this is only possible if the kinetic energy is zero.

$$E_{kin} + U_{eff}(r) = U_{eff,min} \Rightarrow E_{kin} = 0 \tag{2.250}$$

This implies that $\dot{r} = 0$:

$$E_{kin} = \frac{1}{2}m\dot{r}^2 = 0 \rightarrow \dot{r} = 0 \tag{2.251}$$

At first glance, this seems strange: $\dot{r} = 0$ suggests that the earth doesn't move, it has zero velocity. That would indeed be strange: after all we are dealing here with a planet that is attracted via gravity towards the sun. How can it possible have zero velocity?

We are about to make a mistake: $\dot{r} = 0$ doesn't mean that the velocity is zero. It means that $r(t) = const$. The planet still has a velocity perpendicular to its position vector $\vec{r}$. Earlier we found: $l = mrv_\perp = const$. We now have, since

$$\dot{r} = 0 \rightarrow r = r_0 = const, l = mr_0v_\perp = const \rightarrow v_\perp = \frac{l}{m_{r_0}} = const \tag{2.252}$$

Thus, if a planet orbits its sun such that its (pseudo-)potential $U_{eff} = minimum$, then its orbit is a circle of radius $r_0$ that corresponds to the minimum in $U_{eff}$ and the planet has a velocity that is constant in magnitude $v = \frac{l}{mr_0}$.

**Case 2: Effective potential < Total energy < 0**

Next, we consider a case where the total energy of the planet has a value between the minimum of the curve of the effective potential and 0. Call the value of the energy $E_2$.

From Figure 24 we see that the planet will now be confined to an area where the effective potential is either equal to or smaller than this particular value $E_2$

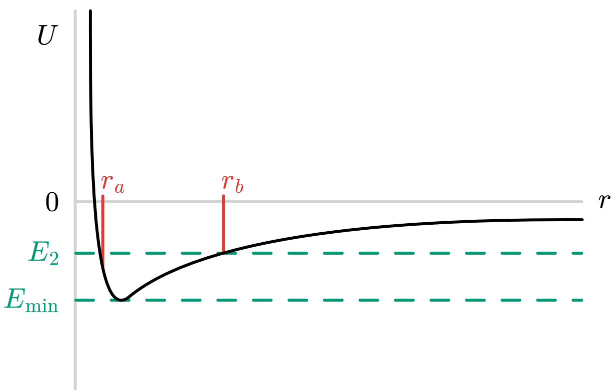


Figure 2.138: Total energy between 0 and minimum of effective potential.

Thus, the trajectory is confined between $r = r_a$ and $r = r_b$. At both these end points, the planet will have zero radial velocity: $\dot{r}_a = \dot{r}_b = 0$. However, as before, the planet will still have angular momentum and thus still have a non-zero velocity. The planet will travel in the $(x, y)$-plane between $r = r_a$ and $r = r_b$. How? We don't know yet.

N.B. Do realize, that the velocity is for this case not a constant. We already have established that it is linked to the angular momentum (which is a constant) and the distance to the origin.

Thus, if the planet is closer to $r_a$ it moves faster than close to $r_b$. But it cannot escape from $r_a < r(t) < r_b$.

**Case 3: Total energy > 0**

Finally, we take the case that the total energy of the planet is positive, say a value of $E_3$ in Figure 25. Now we see that the planet can approach the sun, but not closer than a distance $r = r_c$. The planet is attracted to the sun, but after reaching the closest distance $r = r_c$ it will move away and eventually reach infinity. Again note: at $r = r_c$, the planet does have a non-zero velocity.

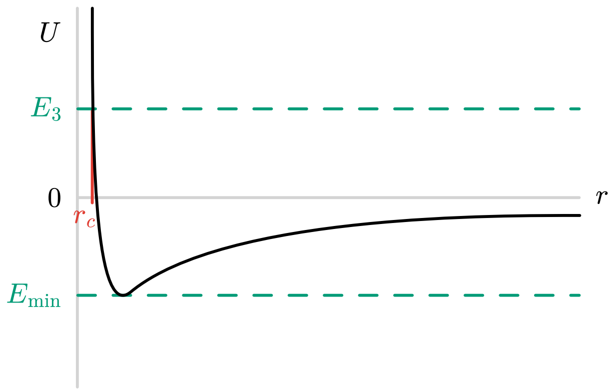



Figure 2.139: Total energy larger than 0.

### 2.4.6.3 Ellipsoidal orbits

We are left with the task of showing that planets 'circle' the sun in an ellipse. From the above, we now know that this must mean that the total energy is smaller than zero: $E < 0$. We will not go over the details of the derivation, but leave that for another course.

The outcome of the analysis would be the following expression for the orbit in case of an ellipse:

$$\frac{(x + ea)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2.253}$$

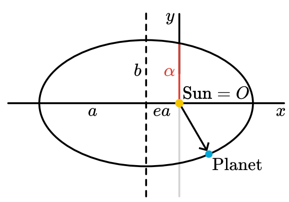



Figure 2.140: Ellips in Cartesian coordinates.

This is an ellipse with semi major and minor-axis $a$ and $b$, respectively. The center of the ellipse is located at $(-ea, 0)$. Note that the sun is in the origin and that seen from the center of the ellipse, the origin is at one of the focal points of the ellipse. Consequently, the orbit is not symmetric as viewed from the sun. We notice this on earth: the summer and winter (when the sun is closest respectively furthest from the sun) are not symmetric, even if we take the tilted axis of the earth into account.

The half and short long axis are given by:

$$a = \frac{\alpha}{1 - e^2} = \frac{GMm}{2 \mid E \mid} \tag{2.254}$$

$$b = a\alpha = \frac{l^2}{2m \mid E \mid} \tag{2.255}$$

with

$$e = \sqrt{1 + \frac{2El^2}{(GMm)^2 m}} \tag{2.256}$$

and

$$\alpha \equiv \frac{l^2}{2GMm^2} \tag{2.257}$$

This type of curve is know as the conic sections. That is, they can be found by intersecting a cone with a plane. See the animation below, where a plane is at various positions and at various angles intersecting a cone.

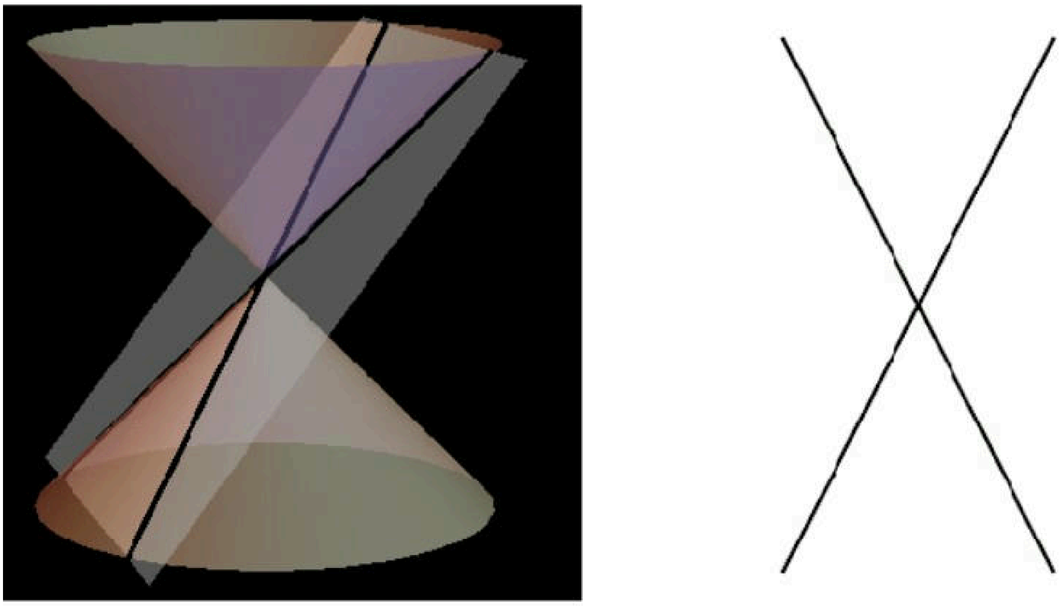


Figure 2.141: Conic sections animation created by Sara van der Werf, used with permission.

Note that in the definition of $e$, the total energy of the system plays a role. This energy can be negative (see Figure 23). The minimum value of the effective potential energy is easily computed. It is $U_{eff,min} = -\frac{1}{2}\frac{(GmM)^2 m}{l^2}$ and is realized when the planet is at a distance $r = \frac{l^2}{GMm^2}$. For this case we have $e = 0$ and the planet is moving in a circle around the sun, as we already argued above.

For $0 \leq e < 1$ the orbit is an ellipse as Kepler already had postulated (for these values of $e$ the orbit is a closed one).

For $e = 1$, the orbit is a parabola: the object will eventually move to infinity where it has exactly zero radial velocity.

Finally, for $e > 1$ the trajectory is a hyperbola with the planet again moving to infinity.

**Conclusion: according to Newton's laws of mechanics, combined with the Gravitation force proposed by Newton, planets must move in ellipses around their star.**

This holds for our solar system, but for any other star with planets as well. Research has shown that there are hundreds of solar systems out in the universe with thousands of planets moving around their star. See e.g. https://exoplanets.nasa.gov/

#### 2.4.6.4 Kepler 3

We are left with proving Kepler's third law:

$$\frac{T_A^2}{R_A^3} = \frac{T_B^2}{R_B^3} = const \tag{2.258}$$

Now that we know the orbit, this is not difficult. We concentrate on the motion during one lapse (one 'year'). From Keppler's $1^st$ law we know that the area a planet sweeps out of its ellipse is given by

$$A(t) = \frac{l}{2m}t + C \tag{2.259}$$

where $C$ is an integration constant. Furthermore, this way of writing makes that the area swept keeps increasing: after one round along the ellipse, we simply keep counting.

However, we can easily back out what happens after exactly one round, or one 'year'. The total area swept is then, of course, the area of the ellipse itself, that is: in one year (time $T$) the area swept is $\pi ab$. Hence we conclude:

$$A(T) = \pi ab \Rightarrow \pi ab = \frac{l}{2m}T \tag{2.260}$$

If we put back what we found for $a$ and $b$, we get

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM} \tag{2.261}$$

Thus, indeed Kepler was right. Moreover, we note that the constant is only depending on the mass of the sun. The same law will hold for other solar systems, but with a different constant.

In Figure 28 Kepler's third law is shown for our solar system. The red data points are based on the measured 'year' of each planet and the distance to the sun. The blue line is the prediction from Newton's theory.

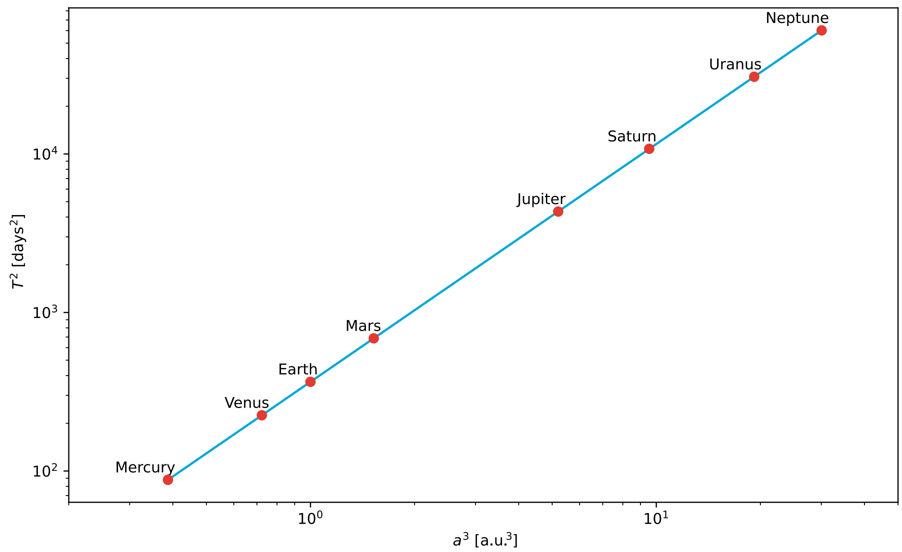



Figure 2.142: Kepler 3 for our solar system.

**Haley's comet**

The planets aren't the only objects that move around the sun. Several icy, rocky smaller objects are trapped in a closed orbit around the sun. These objects, comets from the Greek word for 'long-haired star', are left-overs from when our solar system was formed, some 4.6 billion years ago. There are many comets in our solar system. More than 4500 have been identified, but there are probably much more. Usually the orbit of a comet, if it's a closed one, has a high eccentricity (i.e. close to 1). Moreover, their orbital period may be very long.

One of the best visible comets is Haley's comet. However, its orbital period is about 75 years. It last appeared in the inner parts of the Solar System in 1986. So, you will have to wait until mid-2061 to see it again.

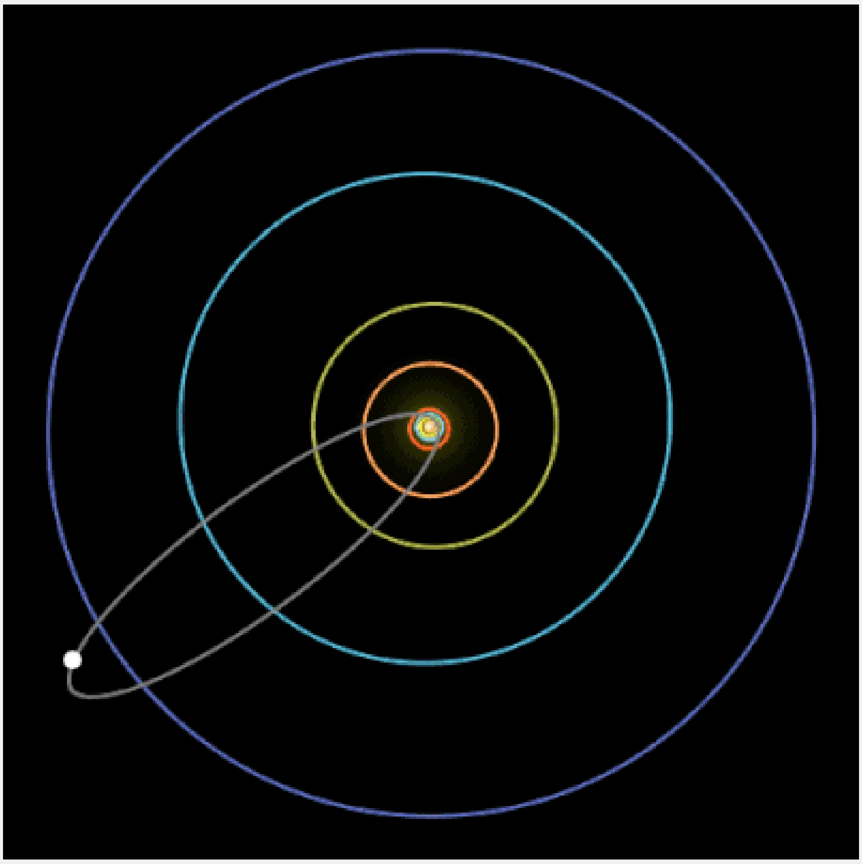


Figure 2.143: Trajectory of Haley's comet. From Wikimedia Commons, licensed under CC-BY 4.0.

### 2.4.7 Speed of the planets & dark matter

Starting from Kepler 3, we can compute the orbital speed of a planet around the sun

$$
\begin{aligned}
T^2 &= \frac{4\pi^2}{GM} a^3 \\
\omega^2 &= \frac{GM}{a^3}, \quad T = \frac{2\pi}{\omega}, \omega = \frac{v}{r}, a \approx r \\
\Rightarrow v &= \sqrt{\frac{GM}{r}}
\end{aligned}
\tag{2.262}
$$

Indeed if we measure the speed of the planets in the solar system this prediction holds, the velocity drops with the distance from the sun as $\propto r^{-1/2}$ (see figure). As $M$ we use the mass of the sun here.

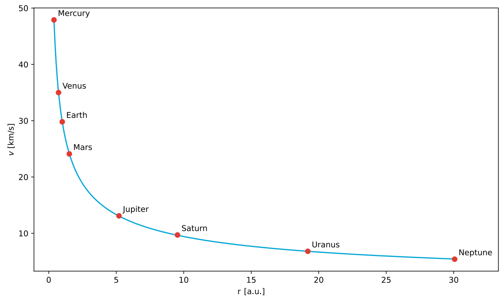



Figure 2.144: From LibreTexts Physics, licensed under CC BY-NC-SA 4.0.

The distance is measured in Astronomical Units [AU], the distance from the earth to the sun (about 8.3 light minutes). Note that the earth is moving with an unbelievable 30 km/s, that is $10^5$ km/h! Do you notice any of that? We will use this motion later with the Michelson-Morley experiment.

If we plot the same speed versus distance curve not for the planets in our solar system, but for stars orbiting the center of our galaxy, the milky way, then the picture looks very different. The far away stars orbit at a much higher speed than expected and the form of the found curve does not match $\propto r^{-1/2}$.

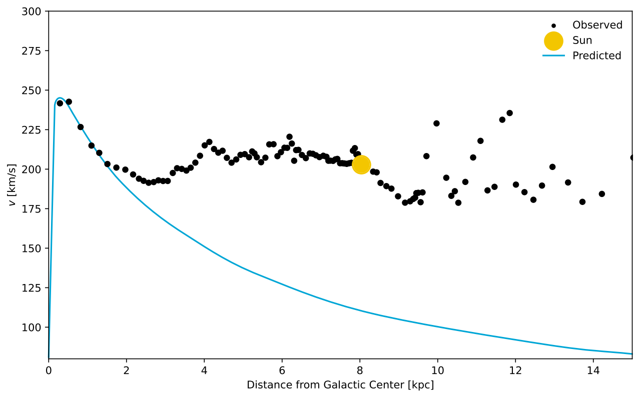



Figure 2.145: Adapted from Wikimedia Commons, licensed under CC-SA 3.0.

This mismatch is not understood to this day! The mass $M$ here is calculated from the visible stars and the supermassive black holes at the center of the galaxy. But even if the mass is calculated wrongly, the shape of the dependency does not match. It turns out, this mismatch is observed in all galaxies! Apparently the law of gravity does not hold for large distances *or* there must be extra mass that increases the speed that we do not see. This mismatch has lead to the postulation of *dark matter* and an *alternative formulation* for the laws of gravity. This is the most disturbing problem in physics today; second is probably the interpretation of measurement in quantum mechanics (collapse of the wave function/Kopenhagen interpretation of Quantum Mechanics; multiverse theories).

The majority of all matter in the universe is believed to be *dark*. And we have no clue what it could be! Most scientist even think it must be non-baryonic, that is, other stuff than our well-known protons or neutrons. It remains most confusing.

The usual distance unit for distances in astronomy outside the solar system is not light years (ly), but parsec [pc], or kpc, or Mpc. One parsec is about 3.3 ly (or $10^{13}$ km). Note: stars visible to the eye are typically not more than a few hundred parsec away. The Milky Way is perfectly visible to the naked eye as a band/stripe of “milk” sprayed over the night sky. But you cannot see it anywhere close to Delft, there is much too much light from cities and greenhouses. Go to Scandinavia in the winter (“wintergatan”) or any place remote where there are few people. The reason you see a “band” in the night sky, is that the Milky Way is a spiral galaxy, sort of pancake shaped, and you see the band in the direction of the pancake.

### 2.4.8 Examples, exercises and solutions

Updated: 13 okt 2025

## 2.5 Conservation Laws / Galilean Transformation

Updated: 20 okt 2025 In the previous chapters, we have seen that from Newton's three laws, we can obtain conservation laws. That means, under certain conditions (depending on the law), a specific quantity cannot change.

These conservation equations are very important in physics. They tell us that no matter what happens, certain quantities will be present in the same amount: they are *conserved.*

Conservation of energy follows from the concept of work and potential energy. Conservation of momentum is a direct consequence of N2 and N3, as we will see below. And finally, under certain conditions, angular momentum is also conserved. In this chapter we will summarize them. The reason is: their importance in physics. These laws are very general and in dealing with physics questions they give guidance and very strict rules that have to be obeyed. They form the foundation of physics that cannot be violated. They provide strong guidance and point at possible directions to look for when analyzing problems in physics.

### 2.5.1 Conservation of Momentum

Consider two particles that mutually interact, that is they exert a force on each other. For each particle we can write down N2:

$$\left.\begin{array}{r}\frac{d\vec{p}_1}{dt} = \vec{F}_{21} \\ \frac{d\vec{p}_2}{dt} = \vec{F}_{12} = -\vec{F}_{21}\end{array}\right\} \rightarrow \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0 \Rightarrow \vec{p}_1 + \vec{p}_2 = const \tag{2.263}$$

The total (linear) momentum is conserved if only internal forces are present; "action-reaction pairs" always cancel out.
This law has no exception: it must be obeyed at all times. The total momentum is constant, momentum lost by one must be gained by others.

### 2.5.2 Conservation of Energy

As we have seen when deriving the concept of potential energy, for a system with conservative forces the total amount of kinetic and potential energy of the system is constant. We can formulate that in a short way as:

$$\sum E_{kin} + \sum V = const \tag{2.264}$$

Again: energy can be redistributed but it cannot disappear nor be formed out of nothing.

If non-conservative forces are present, the right hand side of the equation should be replaced by the work done by these forces.

$$\sum E_{kin} + \sum V = \sum W \tag{2.265}$$

In many cases this will lead to heat, a central quantity in thermodynamics and another form of energy. The "loss" of energy goes always to heat. With this 'generalization' we have a second law that must always hold. Energy cannot be created nor destroyed. All it can do is change its appearance or move from one object to another.

### 2.5.3 Conservation of Angular Momentum

Also angular momentum can be conserved. According to its governing law $\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F}$ it might seem that we can for two interacting particles again invoke N3 "action = -reaction" and the terms with the forces will cancel out. But we need to be a bit more careful, as cross products are involved which are bilinear (a type of mathematical function or operation that is linear in each of two arguments separately, but not necessarily linear when both are varied together). So, let's look at the derivation of "conservation of angular momentum" for two interacting particles:

$$\left.\begin{array}{r}\frac{d\vec{l}_1}{dt} = \vec{r}_1 \times \vec{F}_{21} \\ \frac{d\vec{l}_2}{dt} = \vec{r}_2 \times \vec{F}_{12} = -\vec{r}_2 \times \vec{F}_{21}\end{array}\right\} \rightarrow \frac{d}{dt}\left(\vec{l}_1 + \vec{l}_2\right) = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{21} \tag{2.266}$$

As we see, this is only zero if the vector $\vec{r}_1 - \vec{r}_2$ is parallel to the interaction forces (or zero). We called this a *central force*. Luckily, in many cases the interaction force works over the line connecting the two particles (e.g. gravity). In those cases, the angular momentum is conserved. Mathematically we can write this as:

$$if \vec{F}_{21} \,||\, (\vec{r}_1 - \vec{r}_2) \;\Rightarrow\; \vec{l}_1 + \vec{l}_2 = const \tag{2.267}$$

**Conservation of Mass**

Within the world of Classical Mechanics, mass is also a conserved quantity. Whatever you do, what ever the process the total mass in the system stays the same. We cannot create nor destroy mass. From more modern physics we know that this is not true. On the one hand we can destroy mass. For instance, when an electron and a positron collide, they can annihilate each other resulting in two photons, i.e. 'light particles' that do not have mass. Similarly, we can create mass out of light. This is the inverse of the annihilation: pair production. If we have a photon of at least 1.022 MeV$(= 1.6610^{-13}$J), then -under the right conditions- an electron-positron pair can be created.

Moreover, Albert Einstein showed that mass and energy are equivalent - expressed via his famous equation $E = mc^2$. His theory of Relativity showed us that in collisions at extreme velocities mass is not conserved: it can both be created or disappear. Rephrased: it is actually part of the energy conservation, mass is in that context just a form of energy.

**Emmy Noether, symmetries and conservation laws**

We discussed the conservation laws as consequences of Newton's Laws. That in itself is ok. However, there is a deeper understanding of nature that leads to these conservation laws. And from the conservation laws we can go to Newton's Laws, thus 'reversing the derivations' and starting from this new, different way of looking at nature.

What is it and how do we know? To answer this question we have to resort to Emmy Noether, a German mathematician. Noether made top contributions to abstract algebra. She proved, what we now call, Noether's first and second theorems, which are fundamental in mathematical physics. Noether is often named as one of the best if not the best female mathematicians ever lived. Her work on differential invariants in the calculus of variations has been called "one of the most important mathematical theorems ever proved in guiding the development of modern physics".

Figure 2.146: Amalie Emmy Noether (1882-1935). From Wikimedia Commons, public domain.

Noether shows, that if a dynamic system is invariant under a certain transformation, that is it has a symmetry, then there is a corresponding quantity that is conserved. Ok, pretty abstract. What does it mean, any examples? Yes! Here is one.

In physics we believe that it does not matter if we do an experiment now and repeated it exactly under the same conditions an hour later, the outcome will be the same. Or rephrased: if we translate it in time, the outcome is the same; the laws of physics are invariant. This is in mathematical terms a symmetry, a symmetry with respect to time. Noether's theorem then shows, that there is a conserved quantity and this quantity is energy. Hence, based on the idea that time itself has no effect on physical laws, we immediately arrive at conservation of energy.

Second example: we also believe that place or position in the universe doesn't matter. The physical laws are not only always the same (time invariance), they are also the same everywhere (space invariance). From this symmetry, via Noether's work, we immediately get that momentum is a conserved quantity. Thus, these two invariances or symmetries -time and space - provide us directly with conservation of energy and momentum and from that we could easily derive Newton's second and third law. Much of modern physics is now built on the ideas put forward by Emmy Noether. That goes from quantum mechanics to quarks to string theory.

### 2.5.4 Galilean Transformation

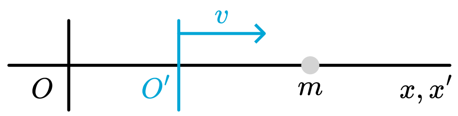
There is one important element of Classical Mechanics that we have to add: for which type of observer do Newton's Laws hold? The original thought was: for inertial observers. These are observers that are at rest with respect to an inertial frame of reference.

But this merely shifts the question to: what is an inertial frame of reference? One possible answer is: an inertial frame of reference is a frame in which Newton's Laws hold. That is: a particle on which, according to an observer in such a frame, no net force is acting will keep moving at a constant velocity.

All inertial frames of reference move at a constant velocity with respect to each other. They cannot accelerate. To picture what it means, an inertial frame of reference or an inertial observer, we sometimes use the idea that such a frame or observer moves at a constant velocity with respect to the 'fixed' stars. And indeed, for a long time people believed that the stars were fixed in space. But from more modern times we do know, that this is not the case: stars are not fixed in space nor do they move at a constant velocity.

Later in the study of Classical Mechanics, we will see, that it is possible to do without the restriction that Newton's Law strictly speaking only hold in inertial frames. But for now, we will stick to inertial frames and look at the 'communication' between two observers in two different inertial frames.

An important requirement of any physical law is that it looks the same for all inertial observers. That doesn't mean that the outcome of using such a law is the same. As a trivial example, take two inertial observers S and S'. According to S, S' moves at a constant velocity, $V$, in the $x$-direction. S' observes a mass $m$ that is not moving in the frame of reference of S'. For simplicity, we will assume that each observer is in its own origin.




S' rightfully concludes, based on Newton's $1^{st}$ law that no force is acting on $m$. S agrees, but doesn't conclude that $m$ is at rest. This is trival: both observers can use Netwon's second law which for this case states that $\frac{d\vec{p}}{dt} = 0 \rightarrow \vec{p} = const \rightarrow \vec{v} = const$. But the constant is not the same in both frames.

To make the above loose statements more precise. We have two coordinates systems CS and CS'. The transformation between both is given by a translation of the origin of S' with respect to that of S.

#### 2.5.4.1 Communication Protocol

We need to have a recipe, a protocol that translates information from $S'$ to $S$ and vice versa.

This protocol is called the *Galilean Transformation* between two inertial frames, $S$ and $S'$.

According to observer $S$, $S'$ is moving at a constant velocity $V$. Both observers have chosen their coordinate system such that $x$ and $x'$ are parallel. Moreover, at $t = t' = 0$, the origins $O$ and $O'$ coincide. The picture below illustrates this.

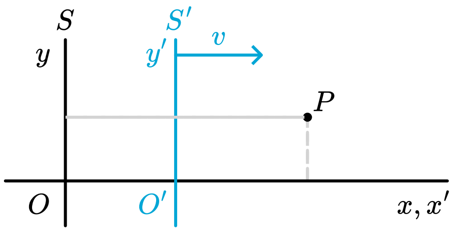



Figure 2.148: Two inertial observers S and S' and their coordinate systems.

Consider for simplicity a 2D point $P$ with coordinates $(x', y')$ and time $t'$ for $S'$. What are the coordinates according to $S$? First of all: in classical mechanics, there is only one time, that is: $t = t'$. Until the days of Einstein this seemed self evident; we now know that nature is more complex.

For the spatial coordinates, we see immediately: $y = y'$. And for the $x$-coordinate $S$ can do the following. To go to the $x$-coordinate of $P$, first $S$ goes to the origin $O'$ of $S'$. $O'$ is a distance $Vt$ from $O$. Thus, the distance to $P$ along the $x$-axis is $Vt + x'$. If we sum the above up, we can formulate the relation between the coordinate system of the two observers. This transformation is the **Galilean Transformation**, or **GT** for short.

**Galilean Transformation**

$$\begin{aligned} x' &= x - Vt \\ y' &= y \\ t' &= t \end{aligned} \tag{2.268}$$

#### 2.5.4.2 Velocity is relative; acceleration is absolute

A direct consequence of the Galilean Transformation is that velocity is observer-dependent (not surprising), but for observers in inertial frames, observed velocities differ by a constant velocity vector.

In what follows we will derive the relations between velocity and acceleration as observed by S and S'. Note that we need to be precise in our notation: $S'$ denotes quantities with a prime (‘), but $S$ does not. This is obvious for the coordinates as $S$ uses $x$ whereas $S'$ will write $x'$. It is, however, also wise to use primes on the velocity: $S$ will denote the $x$-component as: $v_x = \frac{dx}{dt}$. So, $S'$ will note denote velocity by $v$, but by $v'$. Hence $S'$ will write $v'_{x'} = \frac{dx'}{dt'}$. Now, obviously, $t' = t$ so we could drop the prime on time, but it is handy to do that in the second step.

First we look at velocity.

$$\begin{aligned} v'_{x'} &\equiv \frac{dx'}{dt'} \Rightarrow v'_{x'} = \frac{d(x - Vt)}{dt} = v_x - V \\ v'_{y'} &\equiv \frac{dy'}{dt'} \Rightarrow v'_{y'} = \frac{dy}{dt} = v_y \end{aligned} \tag{2.269}$$

Thus indeed velocity is ‘relative’: different observers find different values, but they do have a simple protocol to convert information from the other colleague to their own frame of reference.

Secondly, we look at acceleration.

$$
\begin{aligned}
{a'}_{x'} \equiv \frac{d{v'}_{x'}}{dt'} \Rightarrow {a'}_{x'} = \frac{d(v_x - V)}{dt} = a_x \\
{a'}_{y'} \equiv \frac{d{v'}_{y'}}{dt'} \Rightarrow {a'}_{y'} = \frac{dv_y}{dt} = a_y
\end{aligned}
\tag{2.270}
$$

So, we conclude: acceleration is the same for both observers.

Consequently, N2 holds in both inertial systems if we postulate that $m' = m$. In other words: mass is an object property that does not depend on the observer.

Thus, two observers, each with its own inertial frame of reference, will both *see the same forces*: $F = ma = m'a' = F'$.

This finding is stated as: Newton's second law is *invariant* under Galilean Transformation. Invariant means that the form of the equation does not change if you apply the Galilean coordinate transformation. Later we will expand this to Lorentz invariant transformation in the context of special relativity. The concepts of invariance is very important in physics as hereby we can formulate laws that are the same for everybody (loosely speaking).

### 2.5.5 Exercises, examples & solutions

Updated: 18 jan 2026

#### 2.5.5.1 Worked Example

In class you have seen the *Superballs* example. Let's dive more deep into what is happening.

Figure 2.149: Watch the superballs again.

Consider Figure 2, if you let a smaller and a larger ball drop together, stacked on top of each other, the smaller ball will bounce back much stronger (higher) than if you let the small ball fall without stacking it on the lager ball. How can that happen?

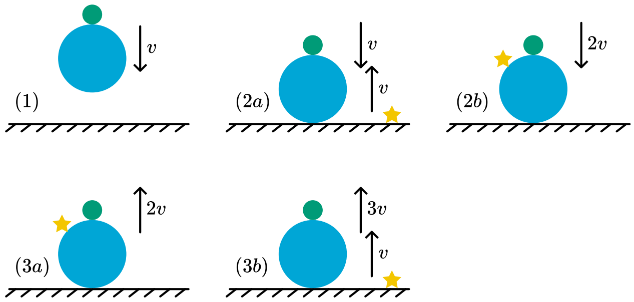



Figure 2.150: Bouncing balls.

To explain this we use the Galilean Transformation (GT). Consider the following situation depicted in Figure 2.

- **1** Both balls are falling with velocity $\vec{v}$ towards the ground.
- **2a** The larger ball just hit the ground. As the mass of the ground is much larger than that of the large ball, it is (elastically) reflected, i.e. the direction of the velocity is reversed but the magnitude stays the same. The small ball is still moving downwards with $\vec{v}$.
- **2b** We apply a GT of the observer (yellow star) from the ground to an observer moving with the larger ball. The observer moving with the larger ball sees the smaller ball moving with $2\vec{v}$ towards it.
- **3a** The smaller ball hits the larger ball and is reflected due to its smaller mass. In the frame of the observer on the larger ball, the smaller ball now moves with $2\vec{v}$ away from it.
- **3b** We apply a GT of the observer (yellow star) from the larger ball back to an observer on the ground. For the observer on the ground the larger ball has velocity $\vec{v}$ upwards from **2a**, therefore the smaller ball has velocity $3\vec{v}$ upwards.

The smaller ball has now velocity $3\vec{v}$ instead of $\vec{v}$ if you drop it without the larger ball. NB: If you would use three balls instead of two, the third ball would have a velocity of $7\vec{v}$ using the same reasoning as above.

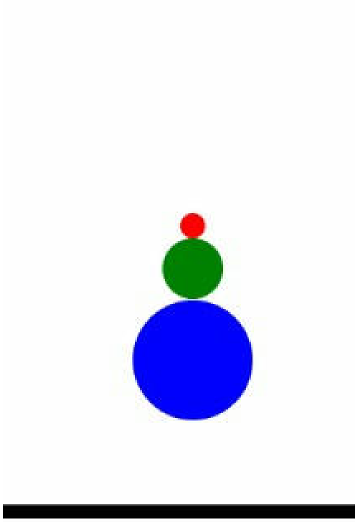


Figure 2.151: Bouncing of three (super)balls.

How much higher does the smaller ball fly with velocity $3\vec{v}$ compared to $\vec{v}$?

**Answer**
We equate the kinetic energy when the ball is just reflected with the potential energy when the ball reached it maximal height before falling back.

$$\frac{1}{2}mv^2 = mgh \Rightarrow h = \frac{v^2}{2g} \tag{2.271}$$

Therefore the ball with $3v$ flies 9 times higher.

**What is very fishy about this whole outcome?**
In situation **1** the kinetic energy is $\frac{1}{2}m_s v^2 + \frac{1}{2}m_\ell v^2$, but in situation **3b** it is $\frac{1}{2}m_s(3v)^2 + \frac{1}{2}m_\ell v^2$ while the potential energy is zero in both cases. This clearly does not add up! But energy must be conserved under all circumstances!

The conclusion is, that we did make an approximation and did not solve the energy and momentum conservation equations for elastic collisions. Even for the case $M \gg m$ there is some momentum transfer. If you solve for the velocity of $m$ after the collision with $M$, you obtain

$$v' = \frac{\frac{m}{M} - 1}{\frac{m}{M} + 1}v \tag{2.272}$$

For $M \gg m$ you indeed see $v' = -v$. Thus the smaller ball will have a smaller velocity than reasoned above *and* the larger ball with also have a smaller velocity (in the experiment you can clearly notice that it does not fly as high as when it drops without the small ball on top). In real life, the balls also deform which makes the collision inelastic.

In a later chapter we will deal with collisions and pay attention to this limit $M \gg m$ in much more detail.

#### 2.5.5.2 Examples

**Example: 8.1**

Consider yourself biking at a constant velocity on an unlikely day with zero wind. Still, you experience a frictional force from the air, with the following observation: the faster you bike, the larger this force. An experimentalist is trying to measure the friction force of the air and relate it to your velocity. She finds that, by and large, these forces turn out to scale with the square of your velocity $v_b$

$$F_f \propto v_b^2 \tag{2.273}$$

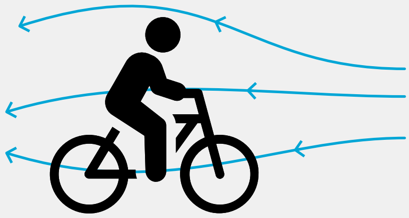


Figure 2.152: Air resistance on cyclist.

Understanding the Galilean transformation, you immediately see that this can't be correct. In your frame of reference, your velocity is zero. And thus, the friction force would be zero. But that cannot be true: both observers should see the same forces. What you see is that the air is blowing at a speed $v_{air} - v_b$ past you. And indeed, the faster you bike, according to the experimentalist, the faster you see the air moving past you: velocity is relative.

You quickly realize that a proper description of the air friction must depend on the relative velocity between you and the air. *Relative* velocities are invariant under Galilean transformation:

$$F_f \propto (v_b - v_{air})^2 \tag{2.274}$$

**Example: 8.2**

Riding a bike while it rains. You have done this hundreds of times. Your front gets soaked, while the backside of your coat stays dry. Or if you have a passenger on your carrier he/she will not get wet, while you take all the water. From a GT to the reference frame of the biker it is obvious why this is the case. The rain is not falling straight from the sky, but at an angle towards him.

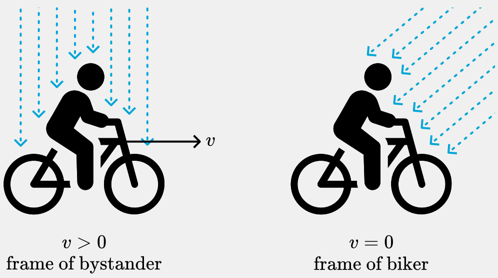



Figure 2.153: Riding a bike in the rain.

NB: For Dutch bikers you have had this experiences with head wind and rain all your life.

### 2.5.5.3 Demo

A ball is bouncing at a wall. The mass of the wall is much greater than that of the ball. So, acceleration of the wall or changes in momentum of the wall can be ignored.

On the left side, we see this from the perspective of an observer, S, standing next to the wall. The right side shows what observer S', who is traveling with the ball as it moves towards the wall, sees. Notice, that both S and S' are inertial observers. That is, they keep their velocity and are no part of the collision.

What would Galilei say?

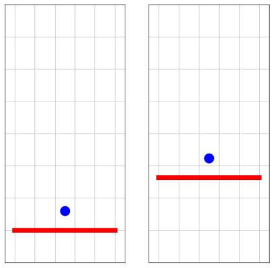


Figure 2.154: Ball bouncing at a wall.

#### 2.5.5.4 Exercises

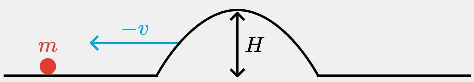
**Exercise 1:** 🌶

A train is passing a station at a constant velocity $V$. At the platform, an observer $S$ sees that in the middle of the train (train length $2L$), at $t = 0$ an object is released with a constant velocity $u$. The object moves towards the back of the train and, at some point in time, will hit the back.


Inside the train, observer $S'$ sees the same phenomenon. Show that both find the same time for the object hitting the back of the train.

**Exercise 2:** 🌶

A point particle of mass $m$ is sitting on a horizontal frictionless table. Gravity is acting in the vertical downward direction.
According to your observation, $m$ has zero velocity. But you see the table moving at a velocity $-v$ in the negative $x$-direction. The table doesn't stay flat, but has a bump of height $H$. What will happen to $m$?




**Exercise 3:** 🌶

Finally, it is winter. And this time, there is lots of fresh snow! You get engaged in a great snowball fight. Your opponent has run out of 'ammunition' and runs away. She is at a distance $L = 2\text{m}$ when she starts running at a speed of $5\text{m/s}$. You throw your last snowball at her at a speed of $10\text{m/s}$.
Determine when and where the snowball hits her. Do that three times:

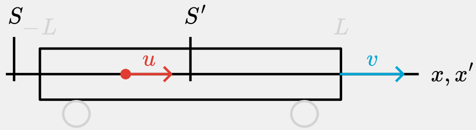
- Your perspective;
- Your opponent's perspective;
- The snowballs perspective.

Next, use the Galilean Transformation and show that you could have used your perspective and GT to find the data for the other two perspectives.

#### 2.5.5.5 Answers

**Solution to Exercise 1:** 🌶

First we make a new sketch, now showing the two observers $S$ and $S'$ and their axis. We have made the velocity of the object red, the color of $S$. And we have given the coordinates of the front and back of the train in grey as these are specified according to $S'$. We do this, as it is is crucial to realize that we have 'mixed' information.




The velocity of the object is $u$ according to $S$. The observer in the train, $S'$, sees a different velocity.
The observer in the train will denote the position of the front of the train by $x_f{}' = L$ and of the back $x_b{}' = -L$. Both are, according to $S'$, fixed values. But $S$ will see that differently.

According to $S'$, the object moves with velocity $u' = u - V$. Note that this is a negative value, otherwise the object will not hit the back of the train.

$S'$ will describe the trajectory of the object by: $x'(t) = x'_0 + u't$ with $x'_0 = 0$. Thus, the object will hit the back of the train at:

$$x'(T') = -L \to u'T' = -L \to T' = \frac{L}{-u'} \tag{2.275}$$

What does $S$ observe? It will write for the trajectory of the object $x_o(t) = ut$ (where we used that the object was released in the middle of the train at $t = 0$ and both observers chose that as their origin).
According to $S$ also the back of the train is moving. It follows a trajectory $x_b = -L + Vt$, since at $t = 0$ the back of the train was at position $x = -L$ according to $S$. The two will collide when

$$x_o(T) = x_b(T) \to uT = -L + VT \to T = \frac{L}{V - u} \tag{2.276}$$

Hence we have $T$ and $T'$ as times of collision. But we already found $u' = u - V$. If we substitute this in $T'$ we get

$$T' = \frac{L}{-u'} = \frac{L}{V - u} = T \tag{2.277}$$

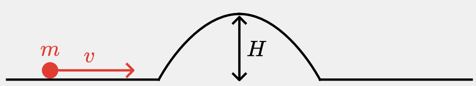
Thus, indeed both observers see the collision at the same moment.

Sneak Preview: much to our surprise, when we enter the world of Special Relativity, this will no longer be the case!

**Solution to Exercise 2:** 🌶

The particle will 'collide' with the bump. This might cause the particle to start moving to the left. How to analyse this situation?

Perhaps it is easier when we view this from the point of view of an observer moving with the table.




Now we have a situation of a particle moving over a friction less table with velocity $v$. If we use conservation of energy, we can write down:

$$\frac{1}{2}mu^2 + mgh = E_0 = \frac{1}{2}mv^2 \tag{2.278}$$

where we have taken $h$ as the height above the table and denote the velocity of $m$ at some point by $u$. The initial height is zero and the initial velocity is $v$.

So, if the initial velocity is such that $\frac{1}{2}mv^2 > mgH$, the particle will go over the bump and come back to height $h = 0$. It will thus pass the bump and then continue moving with velocity $v$. For the original observer this means: the bump will pass the particle and after passing the particle is again laying still (but not at the same position!).

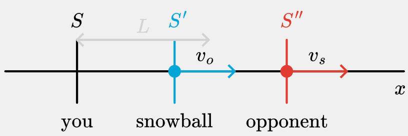
If, on the other hand $v$ is such that $\frac{1}{2}mv^2 < mgH$, the particle will not reach the top of the bump: it has insufficient kinetic energy. In stead it will stop at some height $h^* = \frac{v^2}{2g}$ and then fall of the bump again. It will continue with velocity $-v$ at the flat part of the table. To the original observer this means that $m$ first climbs the bump and returns to get a velocity $-2v$ on the flat part of the table.

The final possibility is $\frac{1}{2}mv^2 = mgH$. In that case the particle will exactly reach the top of the bump and stop there.

N.B. We have assumed that the bump is not too steep, because in such a case the particle will have a real collision with the bump. Think, for instance, of the bump as a sudden step. Then no matter how fast the particle is moving, it will not end up on the step, but bounce back.

**Solution to Exercise 3:**

First, a sketch:




It is a 1-dimensional problem, so an $x$-axis will do. We denote the velocity of your opponent (as seen by you) by $v_o$ and of the snowball $v_s$. The inertial system of you is $S$ and you are sitting in the origin $\mathcal{O}$. Similarly, you opponents inertial system is $S'$ with origin $\mathcal{O}'$ and finally the snowball has inertial system $S$" and the snowball sits in the origin $\mathcal{O}$".

1. Your perspective

$$x_s(t) = v_s t \tag{2.279}$$

$$x_o(t) = L + v_o t \tag{2.280}$$

require: $x_s(t^*) = x_o(t^*)$

$$\rightarrow t^* = \frac{L}{v_s - v_o} = 0.4s \rightarrow x^* = v_s t^* = 4m \tag{2.281}$$

1. Your opponent's perspective

$$v'_s = v_s - v_0 = 5m/s \tag{2.282}$$

require: $x'_s(t'^*) = 0$ since $S'$ is in $x' = 0$. Thus

$$x'_s(t'^*) = -L + v'_s t'^* = 0 \rightarrow t'^* = \frac{L}{v'_s} = 0.4 \tag{2.283}$$

Same time of course. Position: your opponent concludes she is not moving and this she is hit at $x' = 0$.

3. The snowballs perspective.

According to the snowball $v"_o = v_o - v_s = -5m/s$. Thus,

$$x"_o = L + v"_o t \tag{2.284}$$

require: $x"_o(t"^*) = 0$

$$x"_o(t"^*) = L + v"_s t"^* \to t"^* = -\frac{L}{v"_o} = 0.4s \tag{2.285}$$

And, again the snowball will conclude that it all happened in its origin.

Galilean Transformation
We now have three different time/place coordinates for the event 'snowball hits opponent'.

$$\begin{aligned} S : (x_h, t_h) &= (4m, 0.4s) \\ S' : (x'_h, t'_h) &= (0m, 0.4s) \\ S" : (x"_h, t"_h) &= (0m, 0.4s) \end{aligned} \tag{2.286}$$

We could have found this directly from a GT.
a. from $S$ to $S'$: we need to take into account that at $t = 0$ the origins do not coincide. Instead $\mathcal{O}'$ is shifted over a distance L w.r.t. $\mathcal{O}$

$$\begin{aligned} x' &= x - L - v_o t \\ t' &= t \end{aligned} \tag{2.287}$$

Thus: $x'_h = x_h - L - v_o t_h = 0$ and we get indeed $(x'_h, t'_h) = (0\text{m}, 0.4\text{s})$

b. We do a similar exercise for $S$ to $S"$:

$$\begin{aligned} x" &= x - v_s t \\ t" &= t \end{aligned} \tag{2.288}$$

Thus: $x"_h = x_h - v_s t_h = 0$ and we get $(x"_h, t"_h) = (0\text{m}, 0.4\text{s})$

## References


Gribbin, J. (2019). *The scientists: A history of science told through the lives of its greatest inventors*. Random House.

Idema, T. (2023, November 9). *Introduction to particle and continuum mechanics*. https://doi.org/10.59490/tb.81